[Giusti] Minimial Surfaces and Functions of Bounded Variation

Mark Ma

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Contents

Chapter 1

Functions of Bounded Variation

1.1 Functions of Bounded Variation and Caccioppoli Sets

1.1.1 Definitions and Semicontinuity

Definition 1.1.1 (BV Functions)**.** *Let $\Omega \subset \mathbb{R}^n$ be open set. $f \in L^1(\Omega)$.*

$$\int_\Omega |Df| := \sup\{\int_\Omega f \operatorname{div} g \, dx \mid g \in C_0^1(\Omega; \mathbb{R}^n), |g(x)| \le 1\} \tag{1.1}$$

$f \in BV(\Omega)$ if $\int_\Omega |Df| < \infty$. $BV(\Omega)$ is space of $L^1(\Omega)$ functions of bounded variation in Ω.

Example 1.1.1. *If $f \in C^1(\Omega)$, $\int_\Omega |Df| = \int_\Omega |\nabla f| dx$ where $\nabla f \in C(\Omega; \mathbb{R}^n)$ is classical gradient. If $f \in W^{1,1}(\Omega)$, $\int_\Omega |Df| = \int_\Omega |\nabla f| dx$ where $\nabla f \in L^1(\Omega; \mathbb{R}^n)$ is weak gradient.*

Example 1.1.2. *We study $\varphi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \in \mathbb{R}^n \setminus E \end{cases}$ characteristic on E with C^2 boundary.*

- *If E is bounded, $\|\varphi_E\|_{L^1(\Omega)} = |E \cap \Omega| < \infty$ so $\varphi_E \in L^1(\Omega)$. But $\nabla \varphi_E$ distributional derivative is vector-valued Radon measure instead of $L^1(\Omega)$ function, hence $\varphi_E \notin W^{1,1}(\Omega)$. But on the other hand, we may compute $\int_\Omega |D\varphi_E|$. Let $g \in C_0^1(\Omega; \mathbb{R}^n)$ s.t. $|g| \le 1$, so by Gauss-Green formula*

$$\int_\Omega \varphi_E \operatorname{div} g \, dx = \int_E \operatorname{div} g \, dx = \int_{\partial E} g \cdot \nu \, dH_{n-1} \le H_{n-1}(\partial E \cap \Omega) \tag{1.2}$$

for ν outer unit normal to ∂E. Taking supremum in g yields $\int_\Omega |D\varphi_E| < \infty$. Thus $W^{1,1}(\Omega) \subsetneq BV(\Omega)$.

- *We in fact prove $\int_\Omega |D\varphi_E| = H_{n-1}(\partial E \cap \Omega)$. Since E C^2 boundary, $v \in C^1(\partial E; \mathbb{R}^n)$ with $|v| = 1$. Since ∂E is closed in $\mathbb{R}^n$ and $\mathbb{R}^n$ is normal, we may apply Tietze Extension to extend ν to $N \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ with $|N| \le 1$. By Urysohn's there exists $\eta \in C_0^\infty(\Omega)$ s.t. $|\eta| \le 1$, so let $g = \eta N \in C_0^1(\Omega; \mathbb{R}^n)$*

$$\int_\Omega \varphi_E \operatorname{div} g \, dx = \int_E \operatorname{div} g \, dx = \int_{\partial E} \eta N \cdot \nu \, dH_{n-1} = \int_{\partial E} \eta \, dH_{n-1}$$

Take supremum in g on LHS and in η on RHS yields (due to $H_{n-1} \llcorner \partial E$ is Radon measure on $\mathbb{R}^n$)

$$\int_\Omega |D\varphi_E| \ge \sup\{\int_{\partial E} \eta \, dH_{n-1} \mid \eta \in C_0^\infty(\Omega), |\eta| \le 1\} = H_{n-1}(\partial E \cap \Omega) \tag{1.3}$$

Hence (1.2) and (1.3) together gives, for E C^2 boundary

$$\int_\Omega |D\varphi_E| = H_{n-1}(\partial E \cap \Omega) \tag{1.4}$$

Remark 1.1.1. *For $f \in BV(\Omega)$, the duality pairing $\langle Df, g \rangle := -\int_\Omega f \operatorname{div} g \, dx$ defines the distributional gradient $Df \in (C_0^1(\Omega; \mathbb{R}^n))'$ because $\int_\Omega |Df| = \sup_{g \in C_0^1(\Omega; \mathbb{R}^n)} \frac{|\langle Df, g \rangle|}{|g|} < \infty$. By Riesz, the bounded linear functional Df on $C_0^1(\Omega; \mathbb{R}^n)$ defines a vector-valued Radon measure Df on Ω with $\int_\Omega |Df|$ the total variation of Df on Ω. Since $|Df|$ is a Borel measure over Ω, one may measure $\int_A |Df|$ for $A \subset \Omega$ not necessarily open. In particular, if $f = \varphi_E$ for some E bounded and C^2 so that $\varphi_E \in BV(\Omega)$, since the two Borel measures $|D\varphi_E|$ and $H_{n-1} \llcorner \partial E$ agrees on all open sets as in (1.4), they agree on all Borel sets.*

Definition 1.1.2 (Perimeter & Caccioppoli Set)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and E a Borel set. The Perimeter of E in Ω is*

$$P(E, \Omega) := \int_\Omega |D\varphi_E| = \sup\{\int_E \operatorname{div} g \, dx \mid g \in C_0^1(\Omega; \mathbb{R}^n), |g| \le 1\} \tag{1.5}$$

If $\Omega = \mathbb{R}^n$ write $P(E) := P(E, \mathbb{R}^n)$. The Borel set E is a Caccioppoli Set if it has locally finite perimeter, i.e., $P(E, \Omega) < \infty$ for each bounded open $\Omega \subset \mathbb{R}^n$.

Remark 1.1.2. *One has characterisations for Caccioppoli Sets E*

- *E is a Caccioppoli Set iff there exist vector-valued Radon measure ω over $\mathbb{R}^n$ s.t.*

  1. *ω has locally finite variation, i.e., for each bounded open $\Omega \subset \mathbb{R}^n$, $|\omega|(\Omega) < \infty$*
  2. *for all $g \in C_0^1(\Omega; \mathbb{R}^n)$ s.t. $|g| \le 1$, one has $\int_E \operatorname{div} g \, dx = \int g \cdot d\omega$*

  Proof. $\implies$ Since for each Ω bounded and open, $P(E, \Omega) = \int_\Omega |D\varphi_E| < \infty$ iff $\varphi_E \in BV(\Omega)$, $D\varphi_E$ defines a vector-valued Radon measure with locally finite variation over $\mathbb{R}^n$. Let $\omega = -D\varphi_E$, so for each fixed Ω,

  $$\int g \cdot d\omega = -\langle D\varphi_E, g\rangle = \int_\Omega \varphi_E \operatorname{div} g \, dx = \int_E \operatorname{div} g \, dx$$

  $\impliedby$ Suppose such ω exists. Then for any $g \in C_0^1(\Omega; \mathbb{R}^n)$ s.t. $|g| \le 1$

  $$\int_E \operatorname{div} g \, dx = \int g \cdot d\omega \le |\omega|(\Omega) < \infty$$

  take supremum in g on LHS gives $P(E, \Omega) = \int_\Omega |D\varphi_E| \le |\omega|(\Omega) < \infty$. $\square$

- *For E any Borel Set, $\operatorname{supp} D\varphi_E \subset \partial E$ where*

  $$\operatorname{supp} D\varphi_E := \mathbb{R}^n \setminus \bigcup \left\{ A \text{ open} \mid \forall g \in C_0^1(A; \mathbb{R}^n), |g| \le 1 \implies \int g \cdot D\varphi_E = 0 \right\}$$

  Proof. For any $x \notin \partial E$, there exists A open neighbor of x s.t. either $A \subset E$ or $A \subset E^c$. If $A \subset E^c$, $\varphi_E = 0$ on A, so for any $g \in C_0^1(A; \mathbb{R}^n)$, $|g| \le 1$ one indeed has $\int g \cdot D\varphi_E = -\int \varphi_E \operatorname{div} g \, dx = 0$. If $A \subset E$, $\varphi_E = 1$ on A, so for such g, $\int g \cdot D\varphi_E = -\int_E \operatorname{div} g \, dx = -\int \operatorname{div} g \, dx = 0$ since g is compactly supported and one apply the divergence theorem. Thus for any $x \notin \partial E$, $x \notin \operatorname{supp} D\varphi_E$. $\square$

- *E is a Caccioppoli Set iff the Gauss-Green formula holds in a generalized sense, i.e., for any $\Omega \subset \mathbb{R}^n$ open and bounded, and for any $g \in C_0^1(\Omega; \mathbb{R}^n)$ s.t. $|g| \le 1$*

  $$\int_E \operatorname{div} g \, dx = -\int_{\partial E} g \cdot D\varphi_E \tag{1.6}$$

  Proof. $\implies$ follows directly. $\impliedby$ By the previous item, $\int_{\partial E} g \cdot D\varphi_E = \int g \cdot D\varphi_E$. Indeed, $\omega := -D\varphi_E$ has bounded variation on each open bounded Ω. Use the first item that characterises Caccioppoli set. $\square$

- *Given Caccioppoli set E, one has useful identification of $\varphi_E \in BV$*

  Corollary 1.1.1. *For E Caccioppoli, and $\Omega \subset \mathbb{R}^n$ open. If either E or Ω is bounded, $\varphi_E \in BV(\Omega)$.*

  Proof. Since either E or Ω is bounded, $\|\varphi_E\|_{L^1(\Omega)} = |E \cap \Omega| < \infty$ hence $\varphi_E \in L^1(\Omega)$. Now one compute $\int_\Omega |D\varphi_E|$, and may proceed in 2 directions. If Ω itself is bounded, since E Caccioppoli gives locally finite perimeter, indeed $\int_\Omega |D\varphi_E| < \infty$. If on the other hand, E is bounded, for any $g \in C_0^1(\Omega; \mathbb{R}^n)$ s.t. $|g| \le 1$, using (1.6)

  $$\int_\Omega \varphi_E \operatorname{div} g \, dx = \int_E \operatorname{div} g \, dx = -\int_{\partial E} g \cdot D\varphi_E$$

  ∂E is bounded and closed, hence compact. Then one may cover ∂E using sufficient large open ball B_R, and since E is Caccioppoli, $|D\varphi_E|$ defines locally finite variation positive measure

  $$-\int_{\partial E} g \cdot D\varphi_E \le \int_{B_R \cap \Omega} |D\varphi_E| < \infty$$

  $\square$

Theorem 1.1.1 (Semi-continuity). *Let $\Omega \subset \mathbb{R}^n$ open. $\{f_j\} \subset BV(\Omega)$ s.t. $f_j \to f$ in $L^1_{loc}(\Omega)$, then*

$$\int_\Omega |Df| \leq \liminf_{j\to\infty} \int_\Omega |Df_j| \qquad (1.7)$$

Proof. For any $g \in C^1_0(\Omega; \mathbb{R}^n)$ s.t. $|g| \leq 1$

$$\int_\Omega f \operatorname{div} g\, dx = \lim_{j\to\infty} \int_\Omega f_j \operatorname{div} g\, dx \leq \liminf_{j\to\infty} \int_\Omega |Df_j|$$

take supremum in g on LHS. $\qquad\square$

Remark 1.1.3. *The equality in* (1.7) *may not be achieved. Let $\Omega = (0, 2\pi)$ and $f_j(x) = \frac{1}{j}\sin(jx)$. Note $\int_0^{2\pi} |\frac{1}{j}\sin(jx)|dx \leq 2\pi \frac{1}{j} \to 0$ so $f_j \to 0$ in $L^1(0, 2\pi)$. But $f'_j(x) = \cos(jx)$ and $\int_0^{2\pi} |Df_j| = \int_0^{2\pi} |\cos(jx)|dx = 4$.*

Example 1.1.3 (Rational Balls). *One has some very interesting example set E, whose boundary $|\partial E| = H_{n-1}(\partial E) = \infty$ yet $P(E) = \int |D\varphi_E| < \infty$. In particular, this tells us $\int |D\varphi_E| = H_{n-1}(\partial E)$ is not true for non-smooth boundaries. We construct the rational balls $E := \bigcup_{i=0}^\infty B_i$ s.t.*

$$B_i := B_{\frac{1}{2^i}}(x_i) \text{ where } \{x_i\} \text{ truncates all rationals in } \mathbb{R}^n$$

one may first calculate the measure of E

$$|E| \leq \sum_i |B_i| = \sum_i \omega_n \left(\frac{1}{2^i}\right)^n = \frac{\omega_n}{1 - 2^{-n}} < \infty$$

Then since $\overline{E} = \mathbb{R}^n$, we conclude $|\partial E| = H_{n-1}(\partial E) = \infty$. But on the other hand let $E_k = \bigcup_{i=0}^k B_i$ so $\varphi_{E_k} \to \varphi_E$ in $L^1(\mathbb{R}^n)$. Notice E_k has piecewise smooth boundary so $H_{n-1}(\partial E_k) = \int |D\varphi_{E_k}|$, and moreover

$$H_{n-1}(\partial E_k) \leq \sum_{i=0}^k H_{n-1}(\partial B_i) = \sum_{i=0}^k n\omega_n \left(\frac{1}{2^i}\right)^{n-1} \leq n\omega_n \frac{1}{1 - 2^{1-n}} < \infty \qquad uniformly\ in\ k$$

By semicontinuity

$$P(E) = \int |D\varphi_E| \leq \liminf_{k\to\infty} \int |D\varphi_{E_k}| = \liminf_{k\to\infty} H_{n-1}(\partial E_k) \leq n\omega_n \frac{1}{1 - 2^{1-n}} < \infty$$

So E is a Caccioppoli Set.

Proposition 1.1.1. *For $\Omega \subset \mathbb{R}^n$ open, $BV(\Omega)$ with norm $\|f\|_{BV} := \|f\|_{L^1} + \int_\Omega |Df|$ is a Banach Space.*

Proof. That $\|f\|_{BV}$ defines a norm follows from L^1 norm and homogeneity, subadditivity of total variation. To see $BV(\Omega)$ is complete, take Cauchy sequence $\{f_j\}$ in $BV(\Omega)$. Since $\{f_j\}$ is already Cauchy in $L^1(\Omega)$, there exists $f \in L^1(\Omega)$ s.t. $\|f - f_j\|_{L^1} \to 0$. Also, there exits N s.t. $\forall m, n \geq N$, $\int_\Omega |D(f_m - f_n)| \leq 1$, one has $\int_\Omega |Df_j| \leq \max_{1\leq i\leq N} \int_\Omega |Df_i| + 1$ uniformly bounded. Hence (1.7) semicontinuity gives $\int_\Omega |Df| < \infty$ so $f \in BV(\Omega)$. It suffices to show $\int_\Omega |D(f - f_j)| \to 0$. For any $\varepsilon > 0$, there exists N s.t. for any $j, k \geq N$, $\int_\Omega |D(f_j - f_k)| \leq \varepsilon$. Fix j, apply (1.7) semicontinuity to $\{f_j - f_k\}_k$ so $\int_\Omega |D(f_j - f)| \leq \liminf_{k\to\infty} \int_\Omega |D(f_j - f_k)| \leq \varepsilon$. Take ε to 0. $\quad\square$

Proposition 1.1.2. *Let $\Omega \subset \mathbb{R}^n$ open. $f, f_j \in BV(\Omega)$ s.t. $f_j \to f$ in $L^1_{loc}(\Omega)$ and $\int_\Omega |Df| = \lim_{j\to\infty} \int_\Omega |Df_j|$.*

Then for any $A \subset \Omega$ open, one has certain reverse direction to (1.7)

$$\int_{\overline{A}\cap\Omega} |Df| \geq \limsup_{j\to 0} \int_{\overline{A}\cap\Omega} |Df_j|$$

in particular, if $\int_{\partial A\cap\Omega} |Df| = 0$, one has

$$\int_A |Df| = \lim_{j\to 0} \int_A |Df_j| \qquad (1.8)$$

Proof. Let $B := \Omega \setminus \overline{A}$ so $B \subset \Omega$ open. By semicontinuity (1.7)

$$\int_A |Df| \leq \liminf_{j\to 0} \int_A |Df_j| \qquad \int_B |Df| \leq \liminf_{j\to 0} \int_B |Df_j|$$

one calculate

$$\int_{\overline{A}\cap\Omega} |Df| + \int_B |Df| = \int_\Omega |Df| = \lim_{j\to\infty} \int_\Omega |Df_j|$$

$$\geq \limsup_{j\to 0} \int_{\overline{A}\cap\Omega} |Df_j| + \liminf_{j\to\infty} \int_B |Df_j| \geq \limsup_{j\to 0} \int_{\overline{A}\cap\Omega} |Df_j| + \int_B |Df|$$

since $f \in BV(\Omega)$, indeed $\int_B |Df| < \infty$ so one may cancel out. To see (1.8), one notice $A \subset \Omega$. $\qquad\square$

1.1.2   Approximation by smooth functions

Definition 1.1.3. *$\eta(x)$ is mollifier if* $\begin{cases} \eta \in C_0^\infty(\mathbb{R}^n) \\ \operatorname{supp}\eta \subset B_1 \\ \int \eta\, dx = 1 \end{cases}$ *If moreover,* $\begin{cases} \eta \geq 0 \\ \eta(x) = \mu(|x|) \end{cases}$ *η is positive symmetric.*

Standard example for such positive symmetric mollifier is $\eta = \frac{1}{\int \gamma dx}\gamma$ *where* $\gamma(x) := \begin{cases} 0 & |x| \geq 1 \\ \exp(\frac{1}{|x|^2 - 1}) & |x| < 1 \end{cases}$

Definition 1.1.4. *Given a positive symmetric mollifier η, the rescaled mollifier $\eta_\varepsilon(x) := \frac{1}{\varepsilon^n}\eta(\frac{x}{\varepsilon})$ satisfies* $\operatorname{supp}\eta_\varepsilon \subset B_\varepsilon$. *Given $f \in L^1_{loc}(\Omega)$, define its mollification $f_\varepsilon := \eta_\varepsilon * f$*

$$f_\varepsilon(x) = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\eta(\frac{x-y}{\varepsilon})f(y)\, dy = (-1)^n\int_{\mathbb{R}^n}\eta(z)f(x-\varepsilon z)\, dz = \int_{\mathbb{R}^n}\eta(z)f(x+\varepsilon z)\, dz \qquad (1.9)$$

Lemma 1.1.1. *One has tools from mollification*

- *$f_\varepsilon \in C^\infty(\mathbb{R}^n)$, $f_\varepsilon \to f$ in $L^1_{loc}(\Omega)$. If $f \in L^1(\Omega)$, $f_\varepsilon \to f$ in $L^1(\Omega)$.*

- *If $A \leq f(x) \leq B$ for any $x \in \Omega$, then $A \leq f_\varepsilon(x) \leq B$ for any $x \in \Omega$.*

- *If $f, g \in L^1(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} f_\varepsilon g dx = \int_{\mathbb{R}^n} f g_\varepsilon dx$.*

- *If $f \in C^1(\mathbb{R}^n)$, then $(\frac{\partial}{\partial x_i}f)_\varepsilon = \frac{\partial}{\partial x_i}(f_\varepsilon)$ for $i = 1, \cdots, n$.*

- *$\operatorname{supp}f := \overline{\{x \in \mathbb{R}^n \mid f \neq 0\}} \subset A$, then $\operatorname{supp}f_\varepsilon \subset A_\varepsilon := \{x \mid \operatorname{dist}(x, A) \leq \varepsilon\}$.*

Proposition 1.1.3. *$\Omega \subset \mathbb{R}^n$ open, $f \in BV(\Omega)$. For $A \subset\subset \Omega$ open s.t. $\int_{\partial A}|Df| = 0$, one has*

$$\int_A |Df| = \lim_{\varepsilon \to 0}\int_A |Df_\varepsilon|dx \qquad (1.10)$$

Proof. Since $f \in L^1(\Omega)$, $f_\varepsilon \to f$ in $L^1(\Omega)$, by semicontinuity (1.7), one has $\int_A |Df| \leq \liminf_{\varepsilon \to 0}\int_A |Df_\varepsilon|$. It suffices to prove $\int_A |Df| \geq \limsup_{\varepsilon \to 0}\int_A |Df_\varepsilon|$. For any $g \in C_0^1(A; \mathbb{R}^n)$ s.t. $|g| \leq 1$, using tools from mollification

$$\int_A f_\varepsilon \operatorname{div}g\, dx = \int_A f(\operatorname{div}g)_\varepsilon\, dx = \int_A f\operatorname{div}(g_\varepsilon)\, dx$$

$|g| \leq 1 \implies |g_\varepsilon| \leq 1$ and $\operatorname{supp}g \subset A \implies \operatorname{supp}g_\varepsilon \subset A_\varepsilon$. Hence taking supremum in g

$$\int_A |Df_\varepsilon| \leq \int_{A_\varepsilon}|Df|$$

Take $\limsup$ on LHS and use continuity from above on RHS ($f \in BV(\Omega)$ defines a Radon measure $|Df|$)

$$\limsup_{\varepsilon \to 0}\int_A |Df_\varepsilon| \leq \lim \int_{A_\varepsilon}|Df| = \int_{\overline{A}}|Df|$$

Now by our assumption, RHS equals $\int_A |Df|$. $\qquad \square$

Remark 1.1.4. *Note in (1.10) we require $A \subset\subset \Omega$ not because we need boundedness, but because we wish that A and A_ε do not touch $\partial\Omega$. And this problem is resolved for taking $\Omega = \mathbb{R}^n$, and indeed, one may do so for $A = A_\varepsilon = \mathbb{R}^n$ ($\partial A = \partial\mathbb{R}^n = \varnothing$). Now for any $f \in BV(\mathbb{R}^n)$, one has*

$$\int_{\mathbb{R}^n}|Df| = \lim_{\varepsilon \to 0}\int_{\mathbb{R}^n}|Df_\varepsilon|\, dx \qquad (1.11)$$

Indeed for E bounded Caccioppoli, $\varphi_E \in BV(\mathbb{R}^n)$ by Corollary 1.1.1, so (1.11) applies to φ_E.

(1.10) motivates our approximation of $f \in BV(\Omega)$ using smooth functions. Note approximation in BV norm should not be expected since the BV-closure of $C^\infty(\Omega)$ is $W^{1,1}(\Omega) \subsetneq BV(\Omega)$.

Theorem 1.1.2 (Approximation using C^∞). *$\Omega \subset \mathbb{R}^n$ open, $f \in BV(\Omega)$. There exists $\{f_j\} \subset C^\infty(\Omega)$ s.t.*

$$\lim_{j \to \infty}\int_\Omega |f_j - f|\, dx = 0 \qquad (1.12)$$

$$\lim_{j \to 0}\int_\Omega |Df_j|dx = \int_\Omega |Df| \qquad (1.13)$$

Proof. Since $f \in BV(\Omega)$, $|Df|$ on Ω is finite measure, so $\forall \, \varepsilon > 0$, there exists $m \in \mathbb{N}$ s.t. $\int_{\Omega \setminus \Omega_0} |Df| < \varepsilon$ where

$$\Omega_k := \left\{ x \in \Omega \mid \text{dist}(x, \partial\Omega) > \frac{1}{m+k} \right\} \qquad k \geq 0 \tag{1.14}$$

Define sequence $\{A_i\}_{i \geq 1}$ s.t. $A_1 := \Omega_2$, $A_i := \Omega_{i+1} \setminus \overline{\Omega}_{i-1}$ for $i \geq 2$. Note A_i are open and $\Omega \subset \bigcup_{i \geq 1} A_i$. There exists smooth partition of unity $\{\phi_i\}$ subordinate to the cover $\{A_i\}$ s.t.

$$\phi_i \in C_0^\infty(A_i), \quad 0 \leq \phi_i \leq 1, \quad \sum_{i=1}^\infty \phi_i = 1$$

Note for any $x \in \Omega$, at most 2 of the A_i covers x, hence $\sum_i \phi_i$ is finite sum pointwise, thus $f = \sum_{i=1}^\infty f\phi_i$. One wish to construct certain mollification of f so that our desired approximation holds, and a common method is to mollify each $f\phi_i$ with ε_i chose for each $i \geq 1$ then sum them up. Each ε_i needs to satisfy (let $\Omega_{-1} := \varnothing$)

$$\text{supp}(\eta_{\varepsilon_i} * (f\phi_i)) \subset \Omega_{i+2} \setminus \overline{\Omega}_{i-2} \tag{1.15}$$

$$\|\eta_{\varepsilon_i} * (f\phi_i) - f\phi_i\|_{L^1(\Omega)} < \varepsilon/2^i \tag{1.16}$$

$$\|\eta_{\varepsilon_i} * (fD\phi_i) - fD\phi_i\|_{L^1(\Omega)} < \varepsilon/2^i \tag{1.17}$$

and define $f_\varepsilon := \sum_{i=1}^\infty \eta_{\varepsilon_i} * (f\phi_i)$. Note $f_\varepsilon \in C^\infty(\Omega)$ since at each $x \in \Omega$, at most 4 supports from (1.15) covers x, hence finite sum of smooth functions gives smoothness. One immediately has from (1.16)

$$\int_\Omega |f_\varepsilon - f| dx \leq \sum_{i=1}^\infty \int_\Omega |\eta_{\varepsilon_i} * (f\phi_i) - f\phi_i| \, dx < \varepsilon$$

hence (1.12) holds. And by semicontinuity (1.7), one has $\int_\Omega |Df| \leq \liminf_{\varepsilon \to 0} \int_\Omega |Df_\varepsilon|$. It suffices to prove $\int_\Omega |Df| \geq \limsup_{\varepsilon \to 0} \int_\Omega |Df_\varepsilon|$. For any $g \in C_0^1(\Omega; \mathbb{R}^n)$ s.t. $|g| \leq 1$,

$$\int_\Omega f_\varepsilon \text{div} \, g \, dx = \sum_{i=1}^\infty \int_\Omega \eta_{\varepsilon_i} * (f\phi_i) \text{div} \, g \, dx = \sum_{i=1}^\infty \int_\Omega f\phi_i \text{div} \, (\eta_{\varepsilon_i} * g) \, dx$$

notice

$$\text{div}(\phi_i \, \eta_{\varepsilon_i} * g) = D\phi_i \cdot (\eta_{\varepsilon_i} * g) + \phi_i \, \text{div} \, (\eta_{\varepsilon_i} * g)$$

hence

$$\int_\Omega f_\varepsilon \text{div} \, g \, dx = \sum_{i=1}^\infty \int_\Omega f \left[\text{div}(\phi_i \, \eta_{\varepsilon_i} * g) - D\phi_i \cdot (\eta_{\varepsilon_i} * g) \right] dx$$

$$= \int_\Omega f \text{div}(\phi_1 \, \eta_{\varepsilon_1} * g) \, dx + \sum_{i=2}^\infty \int_\Omega f \text{div}(\phi_i \, \eta_{\varepsilon_i} * g) \, dx - \sum_{i=1}^\infty \int_\Omega f D\phi_i \cdot (\eta_{\varepsilon_i} * g) \, dx$$

$$= \int_\Omega f \text{div}(\phi_1 \, \eta_{\varepsilon_1} * g) \, dx + \sum_{i=2}^\infty \int_\Omega f \text{div}(\phi_i \, \eta_{\varepsilon_i} * g) \, dx - \sum_{i=1}^\infty \int_\Omega \eta_{\varepsilon_i} * (fD\phi_i) \cdot g \, dx$$

notice the pointwise finite sum implies

$$\sum_{i=1}^\infty \phi_i = 1 \implies \sum_{i=1}^\infty D\phi_i = 0$$

hence one may add back the sum of gradients

$$\int_\Omega f_\varepsilon \text{div} \, g \, dx = \int_\Omega f \text{div}(\phi_1 \, \eta_{\varepsilon_1} * g) \, dx + \sum_{i=2}^\infty \int_\Omega f \text{div}(\phi_i \, \eta_{\varepsilon_i} * g) \, dx - \sum_{i=1}^\infty \int_\Omega [\eta_{\varepsilon_i} * (fD\phi_i) - fD\phi_i] \cdot g \, dx$$

now by direct estimate, (1.15) and (1.17) respectively

$$\int_\Omega f \text{div}(\phi_1 \, \eta_{\varepsilon_1} * g) \, dx \leq \int_\Omega |Df|$$

$$\sum_{i=2}^\infty \int_\Omega f \text{div}(\phi_i \, \eta_{\varepsilon_i} * g) \, dx \leq 3 \int_{\Omega \setminus \Omega_0} |Df| < 3\varepsilon$$

$$\sum_{i=1}^\infty \int_\Omega [\eta_{\varepsilon_i} * (fD\phi_i) - fD\phi_i] \cdot g \, dx < \varepsilon$$

Hence taking supremum in g on LHS gives

$$\int_\Omega |Df_\varepsilon| \leq \int_\Omega |Df| + 4\varepsilon \implies \limsup_{\varepsilon \to 0} \int_\Omega |Df_\varepsilon| \leq \int_\Omega |Df|$$

and (1.13) immediately follows. $\qquad \square$

Remark 1.1.5 (Boundary Behavior of Smooth Approximation)**.** $\Omega \subset \mathbb{R}^n$ *open,* $f \in BV(\Omega)$. *For every* $\varepsilon > 0$, $N > 0$ *and* $x_0 \in \partial\Omega$, *let* f_ε *be as above*

$$\lim_{\rho \to 0} \frac{1}{\rho^N} \int_{B_\rho(x_0) \cap \Omega} |f_\varepsilon - f| \, dx = 0 \tag{1.18}$$

Proof. For $\varepsilon > 0$, choose $m \in \mathbb{N}$, Ω_k as in (1.14) and f_ε as in Theorem 1.1.2. One wish to determine i_0 w.r.t. ρ so that for any $x \in B_\rho(x_0) \cap \Omega$, one may write

$$f_\varepsilon(x) - f(x) = \sum_{i=1}^{\infty} \left(\eta_{\varepsilon_i} * (f\phi_i) - f\phi_i \right) = \sum_{i=i_0}^{\infty} \left(\eta_{\varepsilon_i} * (f\phi_i) - f\phi_i \right)$$

Making use of (1.15), one needs i_0 to be the smallest integer i s.t. $\partial B_\rho(x_0) \cap \Omega$ touches $\mathrm{supp}\eta_{\varepsilon_i} * (f\phi_i)$, i.e.

$$\frac{1}{m + i_0 + 2} \leq \rho \leq \frac{1}{m + i_0 + 1} \implies i_0 = \lceil \frac{1}{\rho} \rceil - m - 2$$

thus via (1.16), for some constant C independent of ρ

$$\int_{B_\rho(x) \cap \Omega} |f_\varepsilon - f| \, dx \leq \sum_{i=i_0}^{\infty} \|\eta_{\varepsilon_i} * (f\phi_i) - f\phi_i\|_{L^1(\Omega)} \leq C \, 2^{-i_0} = C \, 2^{-\frac{1}{\rho}}$$

where $2^{-\frac{1}{\rho}}$ goes to 0 exponentially fast. Hence multiplying both sides by $\frac{1}{\rho^N}$ and sending $\rho \to 0$ gives (1.18). $\qquad \square$

1.1.3   Compactness Theorem and Existence of Minimizing Caccioppoli sets

One shall recall the GNS type Sobolev Embedding and Rellich Theorem from Sobolev Spaces.

Lemma 1.1.2 (Sobolev Embedding)**.** $\Omega \subset \mathbb{R}^n$ *bounded open.* $\partial\Omega$ *Lipschitz continuous.* $1 \leq p \leq n$. *Then*

$$W^{1,p}(\Omega) \subset L^q(\Omega) \qquad \forall \, 1 \leq q \leq \frac{np}{n-p} \tag{1.19}$$

i.e., for any such $1 \leq q \leq \frac{np}{n-p}$, *there exists* $C = C(n, p, q, \Omega)$ *s.t.*

$$\|f\|_{L^q} \leq C \|f\|_{W^{1,p}} \tag{1.20}$$

Lemma 1.1.3 (Rellich-Kondrachov)**.** $\Omega \subset \mathbb{R}^n$ *bounded open.* $\partial\Omega$ *Lipschitz continuous.* $1 \leq p < n$. *Then*

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega) \qquad \forall \, 1 \leq q < \frac{np}{n-p} \tag{1.21}$$

i.e., each uniformly bounded sequence $\{f_j\}$ *in* $W^{1,p}(\Omega)$ *norm has a convergent subsequence* $\{f_{j_k}\}$ *in* $L^q(\Omega)$ *norm for each* $q \in [1, \frac{np}{n-p})$.

Using above lemmas, one may show for the corresponding BV Embedding and a Compactness Theorem.

Theorem 1.1.3 (GNS-type BV Embedding)**.** $\Omega \subset \mathbb{R}^n$ *bounded open.* $\partial\Omega$ *Lipschitz continuous. Then*

$$BV(\Omega) \subset L^p(\Omega) \qquad \forall \, 1 \leq p \leq \frac{n}{n-1} \tag{1.22}$$

i.e., for any such $1 \leq p \leq \frac{n}{n-1}$, *there exists* $C = C(n, p, \Omega)$ *s.t.*

$$\|f\|_{L^p} \leq C \|f\|_{BV} \tag{1.23}$$

Proof. For any $f \in BV(\Omega)$, by smooth approximation Theorem 1.1.2, choose $\{f_j\} \subset C^\infty(\Omega)$ s.t. $\|f_j - f\|_{L^1} \to 0$ and $\int_\Omega |Df| = \lim_{j \to 0} |Df_j|$. Then there exists M large enough s.t. $\|f_j\|_{BV} \leq M$ uniformly. Since $C^\infty(\Omega) \subset W^{1,1}(\Omega)$, by Sobolev Embedding (1.19), for any $1 \leq p \leq \frac{n}{n-1}$, there exists $C = C(n, p, \Omega)$ s.t.

$$\|f_j\|_{L^p} \leq C \left(\|f_j\|_{L^1} + \|Df_j\|_{L^1} \right) \leq CM$$

uniformly in j. If $p = 1$, by definition of BV norm there's nothing to prove. For $1 < p \leq \frac{n}{n-1}$, the uniform boundedness of f_j in L^p implies, from reflexivity of L^p and Banach Alaoglu, a weakly convergent subsequence in L^p. Still denoting f_j, ones has $f_0 \in L^p$ s.t. $f_j \rightharpoonup f_0$ in L^p. Since Ω is bounded, by Hölder, a priori one knows $f_j, f_0 \in L^1(\Omega)$, and for any $g \in (L^1(\Omega))^* = L^\infty(\Omega)$ (so $g^{\frac{p-1}{p}} \in L^{p'}(\Omega)$)

$$|\int_\Omega (f_j - f_0) \, g| \, dx = |\int_\Omega (f_j - f_0) g^{\frac{p-1}{p}} \, g^{\frac{1}{p}}| \, dx \leq |\int_\Omega (f_j - f_0) g^{\frac{p-1}{p}} \, dx| \, \left\| g^{\frac{1}{p}} \right\|_{L^\infty(\Omega)} \to 0$$

hence one has $f_j \rightharpoonup f_0$ in L^1. But since we already know $f_j \to f$ in L^1, by uniqueness of L^1 strong limit, $f_0 = f$. Finally, by lower semicontinuity of weak convergence,

$$\|f\|_{L^p} \leq \liminf_{j \to 0} \|f_j\|_{L^p} \leq C \liminf_{j \to 0} \left(\|f_j\|_{L^1} + \|Df_j\|_{L^1} \right) = C \|f\|_{BV}$$

$\square$

Theorem 1.1.4 (Compactness). $\Omega \subset \mathbb{R}^n$ *bounded open.* $\partial \Omega$ *Lipschitz continuous. Then*

$$BV(\Omega) \subset\subset L^p(\Omega) \qquad \forall \ 1 \leq p < \frac{n}{n-1} \tag{1.24}$$

i.e., each uniformly bounded sequence $\{f_j\}$ in $BV(\Omega)$ norm has a convergent subsequence $\{f_{j_k}\}$ in $L^p(\Omega)$ norm for each $p \in [1, \frac{n}{n-1})$. Moreover, the limiting function $f \in BV(\Omega)$.

Proof. Let $\{f_j\} \subset BV(\Omega)$ uniformly bounded by $\|f_j\|_{BV(\Omega)} \leq M$. By smooth approximation Theorem 1.1.2, $\forall j$, choose $\tilde{f}_j \in C^\infty(\Omega)$ s.t.

$$\int_\Omega |f_j - \tilde{f}_j| < \frac{1}{j}, \qquad \int_\Omega |D\tilde{f}_j| dx \leq M + 2$$

Now since $\{\tilde{f}_j\} \subset C^\infty(\Omega)$ is uniformly bounded in $W^{1,1}(\Omega)$ norm, by Rellich (1.21), there exists convergent subsequence, still denoting $\tilde{f}_j$, in L^p for any $1 \leq p < \frac{n}{n-1}$. Fix any such p, let $f \in L^p(\Omega)$ s.t. $\left\| \tilde{f}_j - f \right\|_{L^p} \to 0$. Note Ω is bounded, hence Hölder inequality gives convergence in L^1 (p' Hölder conjugate w.r.t p)

$$\int_\Omega |f - \tilde{f}_j| \, dx \leq \left(\int_\Omega |f - \tilde{f}_j|^p \, dx \right)^{\frac{1}{p}} |\Omega|^{\frac{1}{p'}} \to 0$$

and then one may apply semicontinuity (1.7) which gives

$$\int_\Omega |Df| \leq \liminf_{j \to \infty} |D\tilde{f}_j| dx \leq M + 2 < \infty$$

to conclude $f \in BV(\Omega)$. It suffices to show $\|f_j - f\|_{L^p} \to 0$. But by Minkowski

$$\|f_j - f\|_{L^p} \leq \left\| f_j - \tilde{f}_j \right\|_{L^p} + \left\| \tilde{f}_j - f \right\|_{L^p}$$

where the former term convergence due to BV Embedding (1.22) and DCT

$$|f_j - \tilde{f}_j|^p \leq |f_j|^p + |\tilde{f}_j|^p \in L^1(\Omega) \implies \left\| f_j - \tilde{f}_j \right\|_{L^p} \to 0$$

and the latter term converges by Rellich (1.21)

$\square$

Theorem 1.1.5 (Existence of Minimizing Caccioppoli Set). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open, and let L be a Caccioppoli Set. Then there exists a Caccioppoli set E s.t. $E = L$ outside Ω and*

$$\int |D\varphi_E| = \inf\{\int |D\varphi_F| \mid F = L \text{ outside } \Omega\}$$

i.e., $\exists \ E$ Caccioppoli s.t. $E = L$ outside Ω and

$$\int |D\varphi_E| \leq \int |D\varphi_F| \tag{1.25}$$

for any $F \subset \mathbb{R}^n$ Borel s.t. $F = L$ outside Ω.

Proof. One wish to use compactness that extracts a convergent subsequence in L^1. But notice we have no information about regularity of $\partial \Omega$, hence we first take $R > 0$ large s.t. $\Omega \subset\subset B_R(0)$ ball of radius R and we work with B_R. Take a minimizing sequence of sets $\{E_j\}$ s.t. $E_j = L$ outside Ω for any j and

$$\lim_{j \to \infty} \int_{B_R} |D\varphi_{E_j}| = \inf\{\int_{B_R} |D\varphi_F| \mid F = L \text{ outside } \Omega\} \tag{1.26}$$

notice L itself agrees with L outside Ω and since L is a Caccioppoli set, on B_R bounded open, $\int_{B_R} |D\varphi_L| < \infty$. Hence the RHS of (1.26) $< \infty$. Now we may take M large enough so $\int_{B_R} |D\varphi_{E_j}| < M$ uniformly bounded.

And since B_R are bounded, $\varphi_{E_j} \in L^1(B_R)$ for any j, and in particular, $\left\|\varphi_{E_j}\right\|_{L^1(B_R)} \leq |B_R| < \infty$ uniformly, so $\{\varphi_{E_j}\} \subset BV(B_R)$ is uniformly bounded in BV norm. B_R has smooth boundary, so Theorem 1.1.4 gives a convergent subsequence $\varphi_{E_j} \to f$ in $L^1(B_R)$. Again passing to subsequence, $\varphi_{E_j} \to f$ pointwise a.e., but φ_{E_j} are characteristic functions, so $f = \varphi_E$ agrees with characteristic function of some Borel set E a.e. Indeed $E = E_j = L$ outside Ω. And since $\varphi_{E_j} \to \varphi_E$ in $L^1(B_R)$, by semicontinuity (1.7), $\int_{B_R} |D\varphi_E| \leq \lim_{j \to \infty} \int_{B_R} |D\varphi_{E_j}|$

$$\int_{B_R} |D\varphi_E| = \inf\{\int_{B_R} |D\varphi_F| \mid F = L \text{ outside } \Omega\}$$

Finally we recover estimate on $\mathbb{R}^n$ from B_R. For any $F \subset \mathbb{R}^n$ Borel s.t. $F = L$ outside Ω

$$\int |D\varphi_E| = \int_{B_R} |D\varphi_E| + \int_{B_R^c} |D\varphi_E| = \int_{B_R} |D\varphi_E| + \int_{B_R^c} |D\varphi_L|$$

$$\leq \int_{B_R} |D\varphi_F| + \int_{B_R^c} |D\varphi_L| = \int_{B_R} |D\varphi_F| + \int_{B_R^c} |D\varphi_F| = \int |D\varphi_F|$$

Since one may take $F = L$ to be the Caccioppoli Set, E is a Caccioppoli Set.   □

> **Remark 1.1.6.** *One has information for the minimizing set E from Theorem 1.1.5.*
>
> - *L determines boundary values for E. Since $D\varphi_E$ is supported within ∂E, or more particularly, imagine E smooth so $\int_\Omega |D\varphi_E| = H_{n-1}(\partial E \cap \Omega)$ really measures the surface area of ∂E within Ω, then (1.25) indicates that '∂E within Ω' minimizes the surface area for all 'sets **within** Ω that has boundary $\partial L \cap \partial \Omega$'.*
>
> - *Imagine $\partial L \cap \partial \Omega$ fixed, then it determines a surface spanning $\partial L \cap \partial \Omega$. But now curve the portion $\Omega \cap L$ towards Ω, it serves as obstacle forcing '∂E within Ω' away from the minimal surface spanned by $\partial L \cap \partial \Omega$.*

1.1.4   Coarea formula and Smooth Approximation of Caccioppolis sets

One shall recall Coarea formula for Lipschitz functions

Lemma 1.1.4 (Coarea Formula). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz for $n \geq m$. Then for any $A \subset \mathbb{R}^n$ Borel*

$$\int_A \sqrt{\det(Df^* Df)}(x)\, dx = \int_{\mathbb{R}^m} H_{n-m}\left(A \cap f^{-1}(y)\right)\, dy \tag{1.27}$$

With the Classical Coarea formula, one may prove for BV functions.

Theorem 1.1.6 (Coarea Formula). *$\Omega \subset \mathbb{R}^n$ open. $f \in BV(\Omega)$. Denote $F_t := \{x \in \Omega \mid f(x) < t\}$, then*

$$\int_\Omega |Df| = \int_{-\infty}^{\infty} \left(\int_\Omega |D\varphi_{F_t}|\right) dt \tag{1.28}$$

Proof. $\leq$. First let $f \geq 0$. $\forall x \in \Omega$, $f(x) = \int_0^\infty \varphi_{F_t^c}\, dt = \int_0^\infty (1 - \varphi_{F_t})\, dt$, so $\forall g \in C_0^1(\Omega; \mathbb{R}^n)$ s.t. $|g| \leq 1$

$$\int_\Omega f \operatorname{div} g\, dx = \int_\Omega \left(\int_0^\infty (1 - \varphi_{F_t})\, dt\right) \operatorname{div} g\, dx = \int_0^\infty \left(\int_\Omega \operatorname{div} g\, dx - \int_\Omega \varphi_{F_t} \operatorname{div} g\, dx\right) dt$$

By Fubini, and then note compact support of g

$$= -\int_0^\infty \int_\Omega \varphi_{F_t} \operatorname{div} g\, dx\, dt \leq \int_0^\infty \int_\Omega |D\varphi_{F_t}|\, dt$$

Then let $f \leq 0$. $\forall x \in \Omega$, $f(x) = -\int_{-\infty}^0 \varphi_{F_t}\, dt$, so $\forall g \in C_0^1(\Omega; \mathbb{R}^n)$ s.t. $|g| \leq 1$

$$\int_\Omega f \operatorname{div} g\, dx = -\int_\Omega \left(\int_{-\infty}^0 \varphi_{F_t}\, dt\right) \operatorname{div} g\, dx = -\int_{-\infty}^0 \left(\int_\Omega \varphi_{F_t} \operatorname{div} g\, dx\right) dt \leq \int_{-\infty}^0 \int_\Omega |D\varphi_{F_t}|\, dt$$

Hence for any $f \in BV(\Omega)$, write $f = f^+ - f^-$ for $f^+, f^- \geq 0$, so

$$\int_\Omega f \operatorname{div} g\, dx \leq \int_\Omega \left(f^+ - f^-\right) \operatorname{div} g\, dx \leq \int_{-\infty}^\infty \int_\Omega |D\varphi_{F_t}|\, dt$$

taking supremum in g gives $\int_\Omega |Df| \leq \int_{-\infty}^\infty \int_\Omega |D\varphi_{F_t}|\, dt$.

$\geq$. One first show (1.28) for $f \in C(\Omega)$ continuous piecewise linear function. Let $\Omega = \bigcup_{i=1}^N \Omega_i$ for Ω_i disjoint,

open where $f(x) = \langle a_i, x \rangle + b_i$ for $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, $x \in \Omega_i$. Then $\int_\Omega |Df| = \sum_{i=1}^N |a_i||\Omega_i|$. On the other hand, F_t now has piecewise smooth boundary, so

$$\int_{\Omega_i} |D\varphi_{F_t}| = H_{n-1}(\partial F_t \cap \Omega_i) = H_{n-1}\{x \in \Omega_i \mid f(x) = t\} = H_{n-1}\{x \in \Omega_i \mid \langle a_i, x \rangle + b_i = t\}$$

Hence integrating w.r.t. t and by change of coordinates

$$\int_{-\infty}^\infty \int_{\Omega_i} |D\varphi_{F_t}| \, dt = \int_{-\infty}^\infty H_{n-1}\{x \in \Omega_i \mid \langle a_i, x \rangle + b_i = t\} \, dt$$
$$= \int_{-\infty}^\infty |a_i| H_{n-1}\left\{x \in \Omega_i \mid \frac{\langle a_i, x \rangle}{|a_i|} + \frac{b_i}{|a_i|} = \frac{t}{|a_i|}\right\} d\left(\frac{t}{|a_i|}\right)$$
$$= |a_i| \int_{-\infty}^\infty H_{n-1}\left(\Omega_i \cap \left\{\frac{\langle a_i, x \rangle}{|a_i|} + \frac{b_i}{|a_i|} = t\right\}\right) dt$$

using Classical Coarea formula (1.27) with m = 1

$$= |a_i| \int_{\Omega_i} 1 \, dx = |a_i||\Omega_i|$$

hence for $f \in C(\Omega)$ piecewise linear, (1.28) holds

$$\int_\Omega |Df| = \sum_{i=1}^N |a_i||\Omega_i| = \sum_{i=1}^N \int_{-\infty}^\infty \int_{\Omega_i} |D\varphi_{F_t}| \, dt = \int_{-\infty}^\infty \int_\Omega |D\varphi_{F_t}| \, dt$$

Now take any $f \in C^\infty(\Omega)$, approximate using sequence of $\{f_j\} \subset C(\Omega)$ continuous piecewise linear functions in $W^{1,1}(\Omega)$ norm. In particular, one has

$$\|f - f_j\|_{L^1(\Omega)} \to 0, \qquad \|Df\|_{L^1(\Omega)} = \lim_{j \to 0} \|Df_j\|_{L^1(\Omega)} \tag{1.29}$$

where the latter follows from $\|Df - Df_j\|_{L^1(\Omega)} \to 0$ and DCT. Denoting $F_{j,t} := \{x \in \Omega \mid f_j(x) < t\}$, one has

$$|f(x) - f_j(x)| = \int_{-\infty}^\infty |\varphi_{F_t}(x) - \varphi_{F_{j,t}}(x)| \, dt \implies \|f - f_j\|_{L^1(\Omega)} = \int_{-\infty}^\infty \int_\Omega |\varphi_{F_t}(x) - \varphi_{F_{j,t}}(x)| \, dx \, dt \to 0$$

hence there exists a subsequence $\varphi_{F_{j,t}} \to \varphi_{F_t}$ in $L^1(\Omega)$ a.e. t. Since (1.28) holds for each f_j,

$$\int_\Omega |Df| = \lim_{j \to 0} \int_\Omega |Df_j| = \lim_{j \to 0} \int_{-\infty}^\infty \int_\Omega |D\varphi_{F_{j,t}}| \, dt$$

one apply Fatou w.r.t. t

$$\geq \int_{-\infty}^\infty \left(\liminf_{j \to 0} \int_\Omega |D\varphi_{F_{j,t}}|\right) dt$$

then apply semicontinuity (1.7) for BV function

$$\geq \int_{-\infty}^\infty \int_\Omega |D\varphi_{F_t}| \, dt$$

and we conclude (1.28) for $f \in C^\infty(\Omega)$. But notice, we've really only used (1.29) in the above argument. Hence for any $f \in BV(\Omega)$, by Theorem 1.1.2, one may choose $\{f_j\} \subset C^\infty(\Omega)$ s.t. (1.29) holds. Then run the argument again, we conclude (1.28) for $f \in BV(\Omega)$.

$\square$

To show for smooth approximation of sets, one needs Sard's lemma for smooth boundary construction.

Lemma 1.1.5 (Sard's Lemma)**.** $f : \mathbb{R}^n \to \mathbb{R}^m$ C^k *where* $k \geq \max\{n - m + 1, 1\}$. *Let*

$$X := \left\{x \in \mathbb{R}^n \mid Jf(x) := \begin{bmatrix} \nabla f_1 \\ \cdots \\ \nabla f_m \end{bmatrix}(x) \text{ has rank} < m\right\}$$

denote the set of critical points of f. *Then the image* $f(X)$ *has Lebesgue measure* 0 *in* $\mathbb{R}^m$. *In particular, if* $m = 1$, *then given* C^k *map* $f : \mathbb{R}^n \to \mathbb{R}$ *for* $k \geq n$, *one has*

$$\partial\{x \in \mathbb{R}^n \mid f(x) < t\} = \{x \in \mathbb{R}^n \mid f(x) = t\} \quad C^k \text{ boundary for a.e. } t \in \mathbb{R} \tag{1.30}$$

Theorem 1.1.7 (Smooth approximation of Caccioppoli Set). *For $E \subset \mathbb{R}^n$ bounded Caccioppoli set, there exists E_j sets with C^∞ boundary s.t.*

$$\int |\varphi_{E_j} - \varphi_E| dx \to 0 \qquad \int |D\varphi_E| = \lim_{j \to 0} \int |D\varphi_{E_j}| \tag{1.31}$$

Proof. Let η_ε be positive symmetric mollifier. For E Caccioppoli, one look at the mollification $(\varphi_E)_\varepsilon = \eta_\varepsilon * \varphi_E$. Since $(\varphi_E)_\varepsilon$ smooth and compactly supported, indeed $(\varphi_E)_\varepsilon \in BV(\mathbb{R}^n)$. Observe $0 \leq (\varphi_E)_\varepsilon \leq 1$ as inherited from φ_E, and denoting the set $E_{\varepsilon,t} := \{x \in \mathbb{R}^n \mid (\varphi_E)_\varepsilon(x) < t\}$, one has, by Coarea formula (1.28)

$$\int |D(\varphi_E)_\varepsilon| = \int_0^1 \left(\int |D \varphi_{E_{\varepsilon,t}}| \right) dt \tag{1.32}$$

But since E is bounded Caccioppoli, Corollary 1.1.1 gives $\varphi_E \in BV(\mathbb{R}^n)$. One may thus apply global mollification approximation (1.11)

$$\int |D\varphi_E| = \lim_{\varepsilon \to 0} \int |D(\varphi_E)_\varepsilon| = \lim_{\varepsilon \to 0} \int_0^1 \left(\int |D \varphi_{E_{\varepsilon,t}}| \right) dt$$

One now aims for the following claim. One wish to show for any $0 < t < 1$,

$$\int |\varphi_{E^c_{\varepsilon,t}} - \varphi_E| dx \leq \frac{1}{\min\{1 - t, \, t\}} \int |(\varphi_E)_\varepsilon - \varphi_E| dx \tag{1.33}$$

To do so, observe

$$(\varphi_E)_\varepsilon - \varphi_E \geq t \qquad on \ E^c_{\varepsilon,t} \setminus E$$
$$\varphi_E - (\varphi_E)_\varepsilon \geq 1 - t \qquad on \ E \setminus E^c_{\varepsilon,t}$$

Hence

$$\int |(\varphi_E)_\varepsilon - \varphi_E| dx = \int_{E^c_{\varepsilon,t} \setminus E} |(\varphi_E)_\varepsilon - \varphi_E| dx + \int_{E \setminus E^c_{\varepsilon,t}} |(\varphi_E)_\varepsilon - \varphi_E| dx$$

$$\geq t \, |E^c_{\varepsilon,t} \setminus E| + (1 - t) \, |E \setminus E^c_{\varepsilon,t}| \geq \min\{1 - t, \, t\} \int |\varphi_{E^c_{\varepsilon,t}} - \varphi_E| dx$$

which gives (1.33). By mollification, since $\varphi_E \in L^1(\mathbb{R}^n) \subset BV(\mathbb{R}^n)$, $\|(\varphi_E)_\varepsilon - \varphi_E\|_{L^1} \to 0$, hence RHS of (1.33) converges to 0 as $\varepsilon \to 0$ for each t, implying $\left\| \varphi_{E^c_{\varepsilon,t}} - \varphi_E \right\|_{L^1} \to 0$ for each t. But since E bounded, $E^c_{\varepsilon,t} = \{x \mid (\varphi_E)_\varepsilon \geq t\}$ is also bounded for any $0 < t < 1$. And because $\partial E^c_{\varepsilon,t} = \{x \mid (\varphi_E)_\varepsilon = t\}$ is smooth, from example 1.1.2, one has $\varphi_{E^c_{\varepsilon,t}} \in BV(\mathbb{R}^n)$. Hence for $0 < t < 1$, one has semicontinuity (1.7)

$$\liminf_{\varepsilon \to 0} \int |D\varphi_{E^c_{\varepsilon,t}}| \geq \int |D\varphi_E|$$

But because $\mathrm{supp} D\varphi_{E^c_{\varepsilon,t}} \subset \partial E^c_{\varepsilon,t}$, under total variation, one has $\int |D\varphi_{E^c_{\varepsilon,t}}| = \int |D\varphi_{E_{\varepsilon,t}}|$. So

$$\int |D\varphi_E| = \lim_{\varepsilon \to 0} \int |D(\varphi_E)_\varepsilon| = \lim_{\varepsilon \to 0} \int_0^1 \left(\int |D \varphi_{E_{\varepsilon,t}}| \right) dt$$

By Fatou w.r.t. t

$$\geq \int_0^1 \left(\liminf_{\varepsilon \to 0} \int |D\varphi_{E_{\varepsilon,t}}| \right) dt = \int_0^1 \left(\liminf_{\varepsilon \to 0} \int |D\varphi_{E^c_{\varepsilon,t}}| \right) dt \geq \int |D\varphi_E|$$

now combining $\begin{cases} \liminf\limits_{\varepsilon \to 0} \int |D\varphi_{E^c_{\varepsilon,t}}| \geq \int |D\varphi_E| \\ \int_0^1 \left(\liminf\limits_{\varepsilon \to 0} \int |D\varphi_{E^c_{\varepsilon,t}}| \right) dt = \int |D\varphi_E| \end{cases}$ one must have for a.e. $0 < t < 1$

$$\liminf_{\varepsilon \to 0} \int |D\varphi_{E^c_{\varepsilon,t}}| = \int |D\varphi_E|$$

Now one is ready to apply Sard's lemma (1.30) to the set $\partial E^c_{\varepsilon,t} = \{x \in \mathbb{R}^n \mid (\varphi_E)_\varepsilon = t\}$, resulting in smooth boundary of $\partial E^c_{\varepsilon,t}$ for a.e. $0 < t < 1$. Take one such t. we have obtained

$$\begin{cases} \partial E^c_{\varepsilon,t} \quad smooth \\ \left\| \varphi_{E^c_{\varepsilon,t}} - \varphi_E \right\|_{L^1} \to 0 \\ \liminf\limits_{\varepsilon \to 0} \int |D\varphi_{E^c_{\varepsilon,t}}| = \int |D\varphi_E| \end{cases}$$

Take subsequence ε_j s.t. $\varepsilon_j \to 0$ as $j \to \infty$ and $\int |D\varphi_E| = \lim_{j \to 0} \int |D\varphi_{E^c_{\varepsilon_j,t}}|$. Define $E_j := E^c_{\varepsilon_j,t}$. $\qquad \square$

Remark 1.1.7. *Notice E_j bounded and smooth ensures $\varphi_{E_j} \subset BV(\mathbb{R}^n)$, and E bounded Caccioppoli ensures $\varphi_E \in BV(\mathbb{R}^n)$. Hence one may apply* (1.8), *so that for any $A \subset \mathbb{R}^n$ open*

$$\int_A |D\varphi_E| = \lim_{j \to 0} \int_A |D\varphi_{E_j}|$$

1.1.5 Isoperimetric Inequality

One shall first recall from Sobolev Space the GNS inequality as the tool from (1.19) and Poincaré Lemma

Lemma 1.1.6 (GNS Inequality). *$1 \le p < n$. Then there exists $C = C(n, p)$ s.t.*

$$\|f\|_{L^{\frac{np}{n-p}}(\mathbb{R}^n)} \le C \|Df\|_{L^p(\mathbb{R}^n)} \qquad \forall f \in C_0^1(\mathbb{R}^n) \tag{1.34}$$

Lemma 1.1.7 (Poincaré). *$\Omega \subset \mathbb{R}^n$ open, bounded, connected. $\partial\Omega$ Lipschitz continuous. $1 \le p \le \infty$. There there exists $C = C(n, p, \Omega)$ s.t.*

$$\left\| f - \fint_\Omega f \, dy \right\|_{L^p(\Omega)} \le C \|Df\|_{L^p(\Omega)} \qquad \forall f \in W^{1,p}(\Omega) \tag{1.35}$$

Corollary 1.1.2. *There exists $C_1 = C_1(n)$ and $C_2 = C_2(n)$ s.t.*

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le C_1 \|Df\|_{L^1(\mathbb{R}^n)} \qquad \forall f \in C_0^\infty(\mathbb{R}^n) \tag{1.36}$$

$$\|f - f_\rho\|_{L^{\frac{n}{n-1}}(B_\rho)} \le C_2 \|Df\|_{L^1(B_\rho)} \qquad \forall f \in C^\infty(B_\rho) \tag{1.37}$$

where $f_\rho := \fint_{B_\rho} f \, dy = \frac{1}{|B_\rho|} \int_{B_\rho} f \, dy$.

Proof. Apply (1.34) with $p = 1$ yields (1.36). Apply (1.19) with $\Omega = B_\rho$, $p = 1$ and $q = \frac{n}{n-1}$ gives

$$\|f - f_\rho\|_{L^{\frac{n}{n-1}}(B_\rho)} \le C \|f - f_\rho\|_{W^{1,1}(B_\rho)} = C \left(\|f - f_\rho\|_{L^1(B_\rho)} + \|Df\|_{L^1(B_\rho)} \right) \le C_2 \|Df\|_{L^1(B_\rho)}$$

where the last inequality uses (1.35). $\qquad \square$

One immediately has Sobolev Inequalities for BV function.

Theorem 1.1.8 (Sobolev for BV). *There exists $C_1 = C_1(n)$ and $C_2 = C_2(n)$ s.t.*

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le C_1 \int |Df| \qquad \forall f \in BV(\mathbb{R}^n) \text{ and } \operatorname{supp} f \text{ compact} \tag{1.38}$$

$$\|f - f_\rho\|_{L^{\frac{n}{n-1}}(B_\rho)} \le C_2 \int_{B_\rho} |Df| \qquad \forall f \in BV(B_\rho) \tag{1.39}$$

where $f_\rho := \fint_{B_\rho} f \, dy = \frac{1}{|B_\rho|} \int_{B_\rho} f \, dy$.

Proof. One mimic the proof in (1.23). For $f \in BV(\mathbb{R}^n)$ with $\operatorname{supp} f$ compact, by smooth approximation Theorem 1.1.2, there exists $\{f_j\} \subset C_0^\infty(\mathbb{R}^n)$ with uniform compact support s.t. $\|f_j - f\|_{L^1(\mathbb{R}^n)} \to 0$ and $\int |Df| = \lim_{j \to \infty} \int |Df_j| dx$. Now Df_j is uniformly bounded in L^1 on $\mathbb{R}^n$, say by M. So one has from (1.36), $\|f_j\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le C_1 \|Df_j\|_{L^1(\mathbb{R}^n)} \le C_1 M$ uniformly bounded. Since $L^{\frac{n}{n-1}}$ is Reflexive, a uniformly bounded sequence in $L^{\frac{n}{n-1}}$ has a weakly convergent subsequence by Banach Alaoglu, say $f_j \rightharpoonup f_0$ in $L^{\frac{n}{n-1}}$. But with uniform compact support for f_j and f_0, one has $f_j \rightharpoonup f_0$ in L^1 by Hölder. Since we already know $f_j \to f$ in L^1, $f_0 = f$. Now by lower semicontinuity of weak convergence

$$\left(\int |f|^{\frac{n}{n-1}} \, dx \right)^{\frac{n-1}{n}} \le \lim_{j \to \infty} \left(\int |f_j|^{\frac{n}{n-1}} \, dx \right)^{\frac{n-1}{n}} \le C_1 \lim_{j \to \infty} \|Df_j\|_{L^1(\mathbb{R}^n)} = C_1 \int |Df|$$

thus we've proved (1.38). For $f \in BV(B_\rho)$, by smooth approximation Theorem 1.1.2, there exists $\{f_j\} \subset C^\infty(B_\rho)$ s.t. $\|f_j - f\|_{L^1(B_\rho)} \to 0$ and $\int_{B_\rho} |Df| = \lim_{j \to \infty} \int_{B_\rho} |Df_j| dx$, so $\|Df_j\|_{L^1(B_\rho)}$ is uniformly bounded, and by (1.37), $\{f_j - (f_j)_\rho\}$ is uniformly bounded in $L^{\frac{n}{n-1}}(B_\rho)$. Hence there exists weakly convergent subsequence $f_j - (f_j)_\rho \rightharpoonup f_0$ in $L^{\frac{n}{n-1}}(B_\rho)$, thus since B_ρ bounded, $f_j - (f_j)_\rho \rightharpoonup f_0$ weakly in $L^1(B_\rho)$ via Hölder. But $f_j - (f_j)_\rho \to f - f_\rho$ in L^1, so $f - f_\rho = f_0$. Again by the lower semicontinuity one has (1.39)

$$\left(\fint_{B_\rho} |f - f_\rho|^{\frac{n}{n-1}} \, dx \right)^{\frac{n-1}{n}} \le \lim_{j \to \infty} \left(\fint_{B_\rho} |f_j - (f_j)_\rho|^{\frac{n}{n-1}} \, dx \right)^{\frac{n-1}{n}} \le C_2 \lim_{j \to \infty} \|Df_j\|_{L^1(B_\rho)} = C_2 \int_{B_\rho} |Df|$$

$\qquad \square$

Theorem 1.1.9 (Isoperimetric Inequality)**.** *For $E \subset \mathbb{R}^n$ bounded Caccioppoli, there exists $C_1 = C_1(n)$ and $C_2 = C_2(n)$ s.t. for any open ball $B_\rho \subset \mathbb{R}^n$ with radius ρ*

$$|E|^{\frac{n-1}{n}} \leq C_1 \int |D\varphi_E| = C_1 P(E) \tag{1.40}$$

$$\min\{|E \cap B_\rho|, |E^c \cap B_\rho|\}^{\frac{n-1}{n}} \leq C_2 \int_{B_\rho} |D\varphi_E| = C_2 P(E, B_\rho) \tag{1.41}$$

Proof. Since E bounded Caccioppoli, $\varphi_E \in BV(\mathbb{R}^n)$ and $\mathrm{supp}\varphi_E = \overline{E}$ is compact, one apply (1.38) and so (1.40) holds. Now let $f = \varphi_E$, then $f_\rho = \frac{1}{|B_\rho|} \int_{B_\rho} \varphi_E = \frac{|E \cap B_\rho|}{|B_\rho|}$, so

$$\int_{B_\rho} |f - f_\rho|^{\frac{n}{n-1}} \, dx = \int_{B_\rho \cap E} |1 - f_\rho|^{\frac{n}{n-1}} \, dx + \int_{B_\rho \cap E^c} |f_\rho|^{\frac{n}{n-1}} \, dx$$

$$= |B_\rho \cap E| \left(\frac{|E^c \cap B_\rho|}{|B_\rho|} \right)^{\frac{n}{n-1}} + |B_\rho \cap E^c| \left(\frac{|E \cap B_\rho|}{|B_\rho|} \right)^{\frac{n}{n-1}}$$

$$\geq \min\{|B_\rho \cap E|, |B_\rho \cap E^c|\} \left(\left(1 - \frac{|E \cap B_\rho|}{|B_\rho|} \right)^{\frac{n}{n-1}} + \left(\frac{|E \cap B_\rho|}{|B_\rho|} \right)^{\frac{n}{n-1}} \right)$$

Hence taking $\frac{n-1}{n}$ power gives

$$\left(\int_{B_\rho} |f - f_\rho|^{\frac{n}{n-1}} \, dx \right)^{\frac{n-1}{n}} \geq \min\{|B_\rho \cap E|, |B_\rho \cap E^c|\}^{\frac{n-1}{n}} \left(\left(1 - \frac{|E \cap B_\rho|}{|B_\rho|} \right)^{\frac{n}{n-1}} + \left(\frac{|E \cap B_\rho|}{|B_\rho|} \right)^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}}$$

Notice for any $\theta \geq 1$ and $a, b \geq 0$, one has elementary inequality $(a + b)^\theta \leq 2^\theta \left(a^\theta + b^\theta \right)$. Letting $\theta = \frac{n}{n-1}$, $a = 1 - \frac{|E \cap B_\rho|}{|B_\rho|}$ and $b = \frac{|E \cap B_\rho|}{|B_\rho|}$, so

$$\left(\left(1 - \frac{|E \cap B_\rho|}{|B_\rho|} \right)^{\frac{n}{n-1}} + \left(\frac{|E \cap B_\rho|}{|B_\rho|} \right)^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \geq \left(2^{\frac{-n}{n-1}} \cdot 1 \right)^{\frac{n-1}{n}} = \frac{1}{2}$$

independent of size of B_ρ. Hence apply (1.39) one has (1.41). $\qquad \square$

1.2 Traces of BV Function

1.2.1 preliminary lemmas

Lemma 1.2.1 (Lebesgue Differentiation)**.** *$f \in L^1(\mathbb{R}^n)$. Then for a.e. $x \in \mathbb{R}^n$*

$$\lim_{\rho \to 0} \frac{1}{\rho^n} \int_{B_\rho} |f(x + y) - f(x)| \, dy = 0 \tag{1.42}$$

One need Zorn's lemma for a Covering argument.

Lemma 1.2.2 (Zorn's Lemma)**.** *One needs a few definitions to make sense of Zorn's lemma.*

- *A set P is partially ordered by $\leq$ if*

    1. *$\leq$ is reflexive: $x \leq x$ for any $x \in P$*
    2. *$\leq$ is anti-symmetric: $x \leq y$ and $y \leq x$ implies $x = y$*
    3. *$\leq$ is transitive: $x \leq y$ and $y \leq z$ implies $x \leq z$*

    Note not all elements in P are required to be comparable. If a subset $S \subset P$ that inherits the partial order $\leq$ has every pair of elements comparable, S is called totally ordered.

- *An element $m \in P$ with partial order $\leq$ is maximal if there does not exist $s \in P$ s.t. $s \neq m$ and $m \leq s$. Note 'maximal' here does not need m to be comparable with all other elements in P.*

- *Given subset $S \subset P$ that inherits the partial order $\leq$. An element $u \in P$ is an upper bound of S if for any $s \in S$, $s \leq u$.*

Zorn's Lemma claims: Given a nonempty partially order set $(P, \leq)$. If every nonempty subset $S \subset P$ that inherits the order $\leq$ and is totally bounded has an upper bound $u \in P$, then P contains at least one maximal element m with order $\leq$.

Lemma 1.2.3 (Covering Lemma). *$A \subset \mathbb{R}^n$. $\rho : A \to (0, 1)$. Then there exists countable set $\{x_i\} \subset A$ s.t.*

$$B_{\rho(x_i)}(x_i) \cap B_{\rho(x_j)}(x_j) = \varnothing \quad for\ i \neq j \tag{1.43}$$

$$A \subset \bigcup_{i=1}^{\infty} B_{3\rho(x_i)}(x_i) \tag{1.44}$$

Proof. For $k \geq 1$, let $A_k := \{x \in A \mid \frac{1}{2^k} \leq \rho(x) < \frac{1}{2^{k-1}}\}$. One wish to define a sequence of sets L_k for each k. If $A_k = \varnothing$, let $L_k := \varnothing$. WLOG, assume $A_1 \neq \varnothing$. Let $\mathcal{L}_1 := \{L \subset A_1 \mid \forall\, x,\, y \in L,\, x \neq y,\, B_{\rho(x)}(x) \cap B_{\rho(y)}(y) = \varnothing\}$. For nonempty A_1, $\mathcal{L}_1$ is indeed nonempty because both the empty set and singletons are elements of $\mathcal{L}_1$. Now order $\mathcal{L}_1$ with inclusion. For any subcollection of $\mathcal{L}_1$ totally ordered with inclusion, indeed their union is element of $\mathcal{L}_1$ and is upper bounded. Hence $\mathcal{L}_1$ contains a maximal element via Zorn's lemma, call it L_1. Now assume for $L_1, \cdots, L_k$, one obtain L_{k+1} via taking the maximal element of the following collection ordered with inclusion

$$\mathcal{L}_{k+1} := \{L \subset A_{k+1} \mid \forall\, x,\, y \in L_1 \cup L_2 \cup \cdots \cup L_k \cup L,\, x \neq y,\, B_{\rho(x)}(x) \cap B_{\rho(y)}(y) = \varnothing\}$$

Notice $\varnothing \in \mathcal{L}_{k+1}$ is always true so Zorn's lemma applies. L_{k+1} could be empty even if A_{k+1} is nonempty. Moreover, for each L_k, for any $M \subset \mathbb{R}^n$ compact, $M \cap L_k$ must contain finitely many elements otherwise $\{B_{\rho(x)}(x)\}_{x \in M \cap L_k}$ as open cover of $M \cap \overline{L}_k$ does not have finite subcover, contradicting compactness of $M \cap \overline{L}_k$. Hence let M truncate collections of balls $\{\overline{B}_j\}$ with radius $j \in \mathbb{N}$, so each $\overline{B}_j \cap L_k$ is finite for any j. Thus pass j to ∞, L_k is countable. So $L := \bigcup_{k=1}^{\infty} L_k$ is countable set satisfying (1.43). To see (1.44), take any $z \in A = \bigcup_{k=1}^{\infty} A_k$. There must exist k s.t. $z \in A_k$. Now since L_k is maximal element of $\mathcal{L}_k$, $L_k \cup \{z\} \notin \mathcal{L}_k$. Hence there must exist $x \in L_1 \cup \cdots \cup L_k$ s.t. $x \neq z$ and $B_{\rho(x)}(x) \cap B_{\rho(z)}(z) \neq \varnothing$. Note by definition of A_k, $\frac{1}{2^k} \leq \rho(z) < \frac{1}{2^{k-1}}$, and by definition of $L_1 \cup \cdots \cup L_k$, $\frac{1}{2^k} \leq \rho(x) < 1$. Hence $\frac{1}{2}\rho(z) < \rho(x)$. But the balls $B_{\rho(x)}(x) \cap B_{\rho(z)}(z) \neq \varnothing$, so $z \in B_{3\rho(x)}(x)$. $\qquad\square$

Using the covering lemma, one obtains a boundary differentiation lemma analogous to Lemma 1.2.1.

- $B_r(x) := \{z \in \mathbb{R}^n \mid |x - z| < r\}$ ball with center x radius r in $\mathbb{R}^n$

- $\mathscr{B}_\rho(y) := \{t \in \mathbb{R}^{n-1} \mid |y - t| < \rho\}$ ball with center y radius ρ in $\mathbb{R}^{n-1}$

- Let $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_n > 0\}$, $y \in \mathbb{R}^{n-1} = \partial\mathbb{R}^n_+$, $\rho > 0$. Upper cylinder with center y radius and height ρ

$$C^+_\rho(y) := \{(z, t) \in \mathbb{R}^{n-1} \times (0, \infty) \mid |y - z| < \rho,\, 0 < t < \rho\} = \mathscr{B}_\rho(y) \times (0, \rho)$$

Lemma 1.2.4. *μ positive Radon measure on $\mathbb{R}^n_+$ with $\mu(\mathbb{R}^n_+) < \infty$. Then for H_{n-1}-a.e. $y \in \mathbb{R}^{n-1} = \partial\mathbb{R}^n_+$,*

$$\lim_{\rho \to 0} \frac{1}{\rho^{n-1}} \mu(C^+_\rho(y)) = 0 \tag{1.45}$$

Proof. It suffices to show $\forall\, k > 0$, $A_k := \{y \in \mathbb{R}^{n-1} \mid \limsup_{\rho \to 0} \frac{1}{\rho^{n-1}} \mu(C^+_\rho(y)) > \frac{1}{k}\}$ is of H_{n-1} measure zero. Given $\varepsilon > 0$. Note for any $y \in A_k$, there exists $\rho_y < \varepsilon$ s.t.

$$\frac{1}{\rho_y^{n-1}} \mu(C^+_{\rho_y}(y)) > \frac{1}{2k} \iff \rho_y^{n-1} < 2k\, \mu(C^+_{\rho_y}(y))$$

Choose $\{y_j\} \subset A_k$ as in Lemma 1.2.3 with $\rho(y_j) = \rho_{y_j}$ so that $\mathscr{B}_{\rho_{y_j}}(y_j)$ are disjoint and $A_k \subset \bigcup_{j=1}^{\infty} \mathscr{B}_{3\rho_{y_j}}(y_j)$.

$$H_{n-1}(A_k) \leq \sum_{j=1}^{\infty} H_{n-1}(\mathscr{B}_{3\rho_{y_j}}(y_j)) = \omega_{n-1} \sum_{j=1}^{\infty} (3\rho_{y_j})^{n-1} < \omega_{n-1} 3^{n-1} 2k \sum_{j=1}^{\infty} \mu(C^+_{\rho_{y_j}}(y_j))$$

But $C^+_{\rho_{y_j}}(y_j) = \mathscr{B}_{\rho_{y_j}}(y_j) \times (0, \rho_{y_j})$ are disjoint, and since $\rho_{y_j} < \varepsilon$ uniformly in j

$$H_{n-1}(A_k) \leq \omega_{n-1} 3^{n-1} 2k\, \mu\{x \in \mathbb{R}^n_+ \mid 0 < x_n < \varepsilon\}$$

for any $\varepsilon > 0$. But $\mu(\mathbb{R}^n_+) < \infty$, so $\mu\{x \in \mathbb{R}^n_+ \mid 0 < x_n < \varepsilon\} \to 0$ as $\varepsilon \to 0$, hence $H_{n-1}(A_k) = 0$ $\forall\, k > 0$. $\qquad\square$

1.2.2   Existence and Property of Trace on C_R

One first work with upper cylinder $C_R^+ := C_R^+(0) = \mathscr{B}_R \times (0, R)$. Also denote $C_R := \mathscr{B}_R \times (-R, R)$.

Theorem 1.2.1 (Construction of Trace)**.** *$f \in BV(C_R^+)$. There exists $f^+ \in L^1(\mathscr{B}_R)$ s.t. for H_{n-1}-a.e. $y \in \mathscr{B}_R$*

$$\lim_{\rho \to 0} \frac{1}{\rho^n} \int_{C_\rho^+(y)} |f(z) - f^+(y)| \, dz = 0 \tag{1.46}$$

and for any $g \in C_0^1(C_R; \mathbb{R}^n)$, one has

$$\int_{C_R^+} f \operatorname{div} g \, dx = - \int_{C_R^+} \langle g, Df \rangle - \int_{\mathscr{B}_R} f^+ g_n \, dH_{n-1} \tag{1.47}$$

Definition 1.2.1 (Trace of BV Function)**.** *$f \in BV(C_R^+)$. $f^+ \in L^1(\mathscr{B}_R)$ in Theorem 1.2.1 is trace of f on $\mathscr{B}_R$. Indeed (1.46) implies for H_{n-1}-a.e. $y \in \mathscr{B}_R$*

$$f^+(y) = \lim_{\rho \to 0} \frac{1}{|C_\rho^+(y)|} \int_{C_\rho^+(y)} f(z) \, dz \tag{1.48}$$

Proof. First suppose $f \in C^\infty(C_R^+)$. Then for any $0 < \varepsilon < R$, define $f^\varepsilon : \mathscr{B}_R \to \mathbb{R}$ as $f^\varepsilon(y) := f(y, \varepsilon)$. Hence denoting $Q_{\varepsilon', \varepsilon} := \mathscr{B}_R \times (\varepsilon', \varepsilon)$ for $0 \le \varepsilon' < \varepsilon \le R$, one has from FTC

$$\int_{\mathscr{B}_R} |f^\varepsilon(y) - f^{\varepsilon'}(y)| \, dH_{n-1}(y) \le \int_{\mathscr{B}_R} \int_{\varepsilon'}^{\varepsilon} |D_n f(y, t)| \, dt \, dH_{n-1}(y) = \int_{Q_{\varepsilon', \varepsilon}} |D_n f| \, dx \tag{1.49}$$

Since f smooth, RHS Cauchy in ε gives LHS Cauchy in ε, thus $\exists f^+ \in L^1(\mathscr{B}_R)$ s.t. $\|f^\varepsilon - f^+\|_{L^1(\mathscr{B}_R)} \to 0$. Take any $g \in C_0^1(C_R; \mathbb{R}^n)$, Since f smooth, for any $0 < \varepsilon < R$, and let $\nu = (\nu^1, \cdots, \nu^n)$ denote unit normal w.r.t. $\mathscr{B}_R \times \{x_n = \varepsilon\}$ and pointing downwards to $\mathbb{R}^{n-1}$, i.e., $\nu = (0, \cdots, 0, -1)$

$$\begin{aligned}
\int_{Q_{\varepsilon, R}} f \operatorname{div} g \, dx &= - \int_{Q_{\varepsilon, R}} \langle g, Df \rangle + \int_{\mathscr{B}_R \times \{x_n = \varepsilon\}} f(y, \varepsilon) \, g(y, \varepsilon) \cdot \nu \, dH_{n-1}(y) \\
&= - \int_{Q_{\varepsilon, R}} \langle g, Df \rangle - \int_{\mathscr{B}_R \times \{x_n = \varepsilon\}} f(y, \varepsilon) \, g_n(y, \varepsilon) \, dH_{n-1}(y) \\
&= - \int_{Q_{\varepsilon, R}} \langle g, Df \rangle - \int_{\mathscr{B}_R} f^\varepsilon(y) \, g_n^\varepsilon(y) \, dH_{n-1}(y)
\end{aligned}$$

letting $\varepsilon \to 0$, one obtain (1.47) for f smooth. To see for (1.46), for any $y \in \mathscr{B}_R$ and $0 < \rho < R$ s.t. $C_\rho^+(y) \subset C_R^+$

$$\begin{aligned}
\int_{C_\rho^+(y)} |f(z) - f^+(y)| \, dz &= \int_{\mathscr{B}_\rho(y)} \int_0^\rho |f(\eta, t) - f^+(y)| \, dt \, dH_{n-1}(\eta) \\
&\le \int_{\mathscr{B}_\rho(y)} \int_0^\rho |f(\eta, t) - f^+(\eta)| \, dt \, dH_{n-1}(\eta) + \int_{\mathscr{B}_\rho(y)} \int_0^\rho |f^+(\eta) - f^+(y)| \, dt \, dH_{n-1}(\eta) \\
&= \int_{\mathscr{B}_\rho(y)} \int_0^\rho |f(\eta, t) - f^+(\eta)| \, dt \, dH_{n-1}(\eta) + \rho \int_{\mathscr{B}_\rho(y)} |f^+(\eta) - f^+(y)| \, dH_{n-1}(\eta)
\end{aligned}$$

notice upon multiplying by ρ^{-n}, the second term goes to 0 for H_{n-1}-a.e. y due to Lebesgue Differentiation 1.2.1. For the first term, use Fubini and mimic (1.49)

$$\begin{aligned}
\int_{\mathscr{B}_\rho(y)} \int_0^\rho |f(\eta, t) - f^+(\eta)| \, dt \, dH_{n-1}(\eta) &= \int_0^\rho \int_{\mathscr{B}_\rho(y)} |f^t(\eta) - f^+(\eta)| \, dH_{n-1}(\eta) \, dt \\
&\le \int_0^\rho \int_{\mathscr{B}_\rho(y)} \int_0^t |D_n f(\eta, \xi)| \, d\xi \, dH_{n-1}(\eta) \, dt \\
&\le \int_0^\rho \int_{Q_{0,t}(y)} |Df| \, dx \, dt \le \rho \int_{C_\rho^+(y)} |Df|
\end{aligned}$$

now multiplying by ρ^{-n} and notice $|Df|$ is Radon measure on C_R^+ that is finite, one may use (1.45) with $\mu = |Df|$. Hence for H_{n-1}-a.e. $y \in \mathscr{B}_R$

$$\frac{1}{\rho^n} \int_{C_\rho^+(y)} |f(z) - f^+(y)| \, dz \le \frac{1}{\rho^{n-1}} \int_{C_\rho^+(y)} |Df| + \frac{1}{\rho^{n-1}} \int_{\mathscr{B}_\rho(y)} |f^+(\eta) - f^+(y)| \, dH_{n-1}(\eta) \to 0$$

and one concludes (1.46) for f smooth. In general for $f \in BV(C_R^+)$, approximate using $\{f_j\} \subset C^\infty(C_R^+)$ via Theorem 1.1.2. Recall remark (1.18), for any j, given n and H_{n-1}-a.e. $y \in \mathscr{B}_R$

$$\lim_{\rho \to 0} \frac{1}{\rho^n} \int_{C_\rho^+(y)} |f(z) - f_j(z)| \, dz = 0$$

Hence combining with f_j satisfying (1.46)

$$\frac{1}{\rho^n} \int_{C_\rho^+(y)} |f(z) - f_j^+(y)| \, dz \leq \frac{1}{\rho^n} \int_{C_\rho^+(y)} |f(z) - f_j(z)| \, dz + \frac{1}{\rho^n} \int_{C_\rho^+(y)} |f_j(z) - f_j^+(y)| \, dz \to 0$$

for any j. Thus by uniqueness of L^1 limit, all traces f_j^+ coincide H_{n-1}-a.e. $y \in \mathscr{B}_R$. So define $f^+ := f_j^+$ for any such trace. One has (1.46) for $f \in BV(C_R^+)$. Finally, since $\|f - f_j\|_{L^1(C_R^+)} \to 0$ and $\int_{C_R^+} |Df_j| \to \int_{C_R^+} |Df|$, one wish to deduce (1.47) from

$$\int_{C_R^+} f_j \, \mathrm{div} g \, dx = - \int_{C_R^+} \langle g, \, Df_j \rangle - \int_{\mathscr{B}_R} f_j^+ \, g_n \, dH_{n-1}$$

The first term converges due to $\|f - f_j\|_{L^1(C_R^+)} \to 0$ and the last term does not need to converge as $f^+ = f_j^+$ for any j. For the second term, note $\int_{C_R^+} |Df_j| \to \int_{C_R^+} |Df|$ convergence ensures uniform boundedness of $\int_{C_R^+} |Df_j|$. By Banach Alaoglu, the closed unit ball in norm is compact in the weak* topology. Hence identifying $\int_{C_R^+} |Df|$ as norm, there exists subsequence s.t. $Df_j \overset{*}{\rightharpoonup} Df$. But the vague topology convergence $\int_{C_R^+} \langle g, Df_j \rangle \to \int_{C_R^+} \langle g, Df \rangle$ is essentially the weak* topology convergence. Hence we're done.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Proposition 1.2.1 (Approximation in BV implies Approximation in Trace). $f \in BV(C_R^+)$. If $\{f_j\} \subset BV(C_R^+)$ s.t. $f_j \to f$ in $L^1(C_R^+)$ and

$$\lim_{j \to \infty} \int_{C_R^+} |Df_j| = \int_{C_R^+} |Df| \tag{1.50}$$

then

$$\lim_{j \to \infty} \int_{\mathscr{B}_R} |f_j^+ - f^+| \, dH_{n-1}(y) = 0 \tag{1.51}$$

Proof. For any $0 < \beta < R$, consider $Q_{0,\beta} := \mathscr{B}_R \times (0, \beta)$. Define $f_\beta : \mathscr{B}_R \to \mathbb{R}$ s.t. $f_\beta(y) := \frac{1}{\beta} \int_0^\beta f(y,t) \, dt$ for any $f \in BV(C_R^+)$. Then for a.e. β

$$\int_{\mathscr{B}_R} |f^+(y) - f_\beta(y)| \, dH_{n-1}(y) = \int_{\mathscr{B}_R} |f^+(y) - \frac{1}{\beta} \int_0^\beta f(y, t) \, dt| \, dH_{n-1}(y)$$

$$= \frac{1}{\beta} \int_0^\beta \int_{\mathscr{B}_R} |f^+(y) - f(y, t)| \, dH_{n-1}(y) \, dt \; \leq \frac{1}{\beta} \int_0^\beta \int_{Q_{0,t}} |Df| \, dx \, dt \leq \int_{Q_{0,\beta}} |Df| \, dx \tag{1.52}$$

where the last line uses (1.49), initially shown for smooth f. To make sense of (1.49) for $f \in BV(C_R^+)$, one precisely needs smooth approximation from Theorem 1.1.2 where $\|f_\varepsilon \to f\|_{L^1(C_R^+)}$ implies for a.e. t

$$\int_{\mathscr{B}_R} |f_\varepsilon^+(y) - f_\varepsilon(y, t)| \, dH_{n-1}(y) \; \to \; \int_{\mathscr{B}_R} |f^+(y) - f(y, t)| \, dH_{n-1}(y)$$

and $\int_{C_R^+} |Df_\varepsilon| \to \int_{C_R^+} |Df|$ implies via (1.8) ($\int_{\mathscr{B}_R \times \{t\}} |Df| = 0$ for a.e. t otherwise uncountably many disjoint summing up contradicts $f \in BV(C_R^+)$) that $\int_{Q_{0,t}} |Df_\varepsilon| \to \int_{Q_{0,t}} |Df|$. Hence for $\{f_j\} \subset BV(C_R^+)$ as assumed

$$\int_{\mathscr{B}_R} |f_j^+ - f^+| \, dH_{n-1}(y) \leq \int_{\mathscr{B}_R} |f_j^+ - (f_j)_\beta| \, dH_{n-1}(y) + \int_{\mathscr{B}_R} |(f_j)_\beta - f_\beta| \, dH_{n-1}(y) + \int_{\mathscr{B}_R} |f_\beta - f^+| \, dH_{n-1}(y)$$

using (1.52)

$$\leq \int_{Q_{0,\beta}} |Df_j| + \int_{\mathscr{B}_R} |(f_j)_\beta - f_\beta| \, dH_{n-1}(y) + \int_{Q_{0,\beta}} |Df|$$

the middle term writes, using $\|f_j - f\|_{L^1(C_R^+)} \to 0$

$$\int_{\mathscr{B}_R} |(f_j)_\beta - f_\beta| \, dH_{n-1}(y) = \frac{1}{\beta} \int_0^\beta \int_{\mathscr{B}_R} |f_j(y, t) - f(y, t)| \, dH_{n-1}(y) \, dt = \frac{1}{\beta} \int_{C_R^+} |f_j - f| \, dx \to 0$$

Thus, since for a.e. β, $\int_{Q_{0,\beta}} |Df_j| \to \int_{Q_{0,\beta}} |Df|$, one has

$$\limsup_{j\to\infty} \int_{\mathscr{B}_R} |f_j^+ - f^+| \, dH_{n-1}(y) \le 2 \int_{Q_{0,\beta}} |Df|$$

for a.e. β. Thus using $f \in BV(C_R^+)$ so $\int_{Q_{0,\beta}} |Df| \to 0$ as $\beta \to 0$, one arrives at (1.51). $\qquad\square$

Note for $C_R^- := \mathscr{B}_R \times (-R, 0)$, one may similarly define $f^- \in L^1(\mathscr{B}_R)$ as trace for the function $f \in BV(C_R^-)$ via Theorem 1.2.1.

Proposition 1.2.2 (Extension Property for BV)**.** *For $f_1 \in BV(C_R^+)$ and $f_2 \in BV(C_R^-)$, let $f^+, f^- \in L^1(\mathscr{B}_R)$ be their trace respectively. Then for $f : C_R = \mathscr{B}_R \times (-R, R) \to \mathbb{R}$ defined as $f := \begin{cases} f_1 & in\ C_R^+ \\ f_2 & in\ C_R^- \end{cases}$, one has $f \in BV(C_R)$ and*

$$\int_{\mathscr{B}_R} |f^+ - f^-| \, dH_{n-1}(y) = \int_{\mathscr{B}_R} |Df| \tag{1.53}$$

Proof. Note from (1.47) applied to f_1 and f_2 respectively, one has for any $g \in C_0^1(C_R; \mathbb{R}^n)$

$$\int_{C_R^+} f_1 \operatorname{div} g \, dx = -\int_{C_R^+} \langle g, Df_1 \rangle - \int_{\mathscr{B}_R} f^+ g_n \, dH_{n-1}$$

$$\int_{C_R^-} f_2 \operatorname{div} g \, dx = -\int_{C_R^-} \langle g, Df_2 \rangle + \int_{\mathscr{B}_R} f^- g_n \, dH_{n-1}$$

Notice on C_R^-, while deriving (1.47) for smooth f, one take unit normal $\nu = (0, \cdots, 0, 1)$ pointing upwards to $\mathbb{R}^{n-1}$. Hence the last term involving g_n has opposite signs. One take sum of the above to obtain

$$\int_{C_R} f \operatorname{div} g \, dx = -\int_{C_R^+} \langle g, Df_1 \rangle - \int_{C_R^-} \langle g, Df_2 \rangle - \int_{\mathscr{B}_R} (f^+ - f^-) g_n \, dH_{n-1} \tag{1.54}$$

Now if require $|g| \le 1$, one has

$$\left| \int_{C_R} f \operatorname{div} g \, dx \right| \le \int_{C_R^+} |Df_1| + \int_{C_R^-} |Df_2| + \int_{\mathscr{B}_R} |f^+| \, dH_{n-1} + \int_{\mathscr{B}_R} |f^-| \, dH_{n-1} < \infty$$

Hence $f \in BV(C_R)$. But on the other hand, by definition of distributional gradient Df

$$\int_{C_R} f \operatorname{div} g \, dx = -\int_{C_R} \langle g, Df \rangle = -\int_{C_R^+} \langle g, Df \rangle - \int_{C_R^-} \langle g, Df \rangle - \int_{\mathscr{B}_R} \langle g, Df \rangle$$

Notice f coincides with f_1 and f_2 respectively on C_R^+ and C_R^-, hence

$$\int_{C_R} f \operatorname{div} g \, dx = -\int_{C_R^+} \langle g, Df_1 \rangle - \int_{C_R^-} \langle g, Df_2 \rangle - \int_{\mathscr{B}_R} \langle g, Df \rangle \tag{1.55}$$

Now combining (1.54) and (1.55) gives

$$\int_{\mathscr{B}_R} (f^+ - f^-) g_n \, dH_{n-1} = \int_{\mathscr{B}_R} \langle g, Df \rangle$$

so

$$\int_{\mathscr{B}_R} |Df| = \sup_{\substack{g \in C_0^1(C_R; \mathbb{R}^n) \\ |g| \le 1}} \left| \int_{\mathscr{B}_R} \langle g, Df \rangle \right| = \sup_{\substack{g \in C_0^1(C_R; \mathbb{R}^n) \\ |g| \le 1}} \left| \int_{\mathscr{B}_R} (f^+ - f^-) g_n \, dH_{n-1} \right| = \int_{\mathscr{B}_R} |f^+ - f^-| \, dH_{n-1}$$

where the last equality holds by Riesz Representation. Hence we're done with (1.53). $\qquad\square$

1.2.3   Trace on Lipschitz Domains

One has systematic tools to reduce a Domain to C_R. Let $\Omega \subset \mathbb{R}^n$ open with $\partial\Omega$ Lipschitz.

- Since $\partial\Omega$ Lipschitz, for any $x_0 \in \partial\Omega$, there exists a neighborhood around x_0 s.t. the intersection of $\partial\Omega$ and the neighborhood is locally the graph of a Lipschitz function. Due to topology in $\mathbb{R}^n$, one is in fact free to choose the neighborhood as simple geometric objects. Via translation, one may first put $x_0 = 0$ as the origin, then rotate $\partial\Omega$ so that one may choose a cylinder $C(R) = \mathscr{B}_R \times (-\frac{R}{2}, \frac{R}{2})$ with $\mathscr{B}_R$ radius $R > 0$ and height $\frac{R}{2}$, as well as a local Lipschitz function $w : \mathscr{B}_R \subset \mathbb{R}^{n-1} \to (-\frac{R}{2}, \frac{R}{2})$ where the local boundary and interior writes

$$\partial\Omega \cap C(R) = \left\{ (y, t) \in C(R) = \mathscr{B}_R \times (-\frac{R}{2}, \frac{R}{2}) \mid t = w(y) \right\} \tag{1.56}$$

$$\Omega \cap C(R) = \{ (y, t) \in C(R) \mid t > w(y) \} \tag{1.57}$$

- One may further flatten out the local boundary by introducing the variables

$$(y, \tau) = (y, t - w(y)) \in C_R^+ = \mathscr{B}_R \times (0, R)$$

  hence for $f \in BV(\Omega \cap C(R))$, one may further define for $g \in BV(C_R^+)$ via

$$g(y, \tau) := f(y, w(y) + \tau) = f(y, t) \tag{1.58}$$

- Apply Theorem 1.2.1 to $g \in BV(C_R^+)$, there exists trace $g^+ \in L^1(\mathscr{B}_R)$. One define $f^+ \in L^1(\partial\Omega \cap C(R))$ for $f \in BV(\Omega \cap C(R))$ as the trace on local Lipschitz boundary via

$$f^+(y, w(y)) := g^+(y) \tag{1.59}$$

Theorem 1.2.2 (Construction of Trace). *$\Omega \subset \mathbb{R}^n$ open and bounded with $\partial\Omega$ Lipschitz. $f \in BV(\Omega)$. Then there exists trace $\varphi \in L^1(\partial\Omega)$ s.t. for H_{n-1}-a.e. $x \in \partial\Omega$*

$$\lim_{\rho \to 0} \frac{1}{\rho^n} \int_{B_\rho(x) \cap \Omega} |f(z) - \varphi(x)| \, dz = 0 \tag{1.60}$$

And for any $g \in C_0^1(\mathbb{R}^n; \mathbb{R}^n)$ one has, denoting ν outer unit normal w.r.t. $\partial\Omega$

$$\int_\Omega f \operatorname{div} g \, dx = -\int_\Omega \langle g, Df \rangle + \int_{\partial\Omega} \varphi \langle g, \nu \rangle \, dH_{n-1} \tag{1.61}$$

Proof. For $\Omega \subset \mathbb{R}^n$ bounded, $\partial\Omega$ is compact. Hence consider open cover $\{C_x(R)\}_{x \in \partial\Omega}$ where $C_x(R)$ is the cylinder s.t. upon translation and rotation, (1.56) and (1.57) holds for x positioned at the origin. There exists finite subcover $\{C_{x_i}(R_i)\}_{i=1}^N$. Given $f \in BV(\Omega)$, upon defining local trace $f_i^+ \in L^1(\partial\Omega \cap C_{x_i}(R_i))$ for each $f|_{C_{x_i}(R_i)}$ as in (1.59), one observe that on their overlaps they must agree H_{n-1}-a.e. due to uniqueness of L^1 limit. Hence $\varphi(x) := f_i^+(x)$ for i s.t. $x \in C_{x_i}(R_i)$ is a well-defined $L^1(\partial\Omega)$ function. Note for any $x \in \partial\Omega$, and for i s.t. $x \in C_{x_i}(R_i)$, there exists $\rho < \frac{R_i}{2}$ s.t. $B_\rho(x) \subset C_{x_i}(R_i)$. Hence (1.60) follows directly from (1.46) as a local behavior. To derive (1.61), one needs partition of unity. Denote $\Gamma_i := C_{x_i}(R_i)$ for $i \geq 1$ and $\Gamma_0 \subset\subset \Omega$ chosen s.t. $\overline{\Omega} \subset \bigcup_{i=0}^N \Gamma_i$ is open cover. One may choose a smooth partition of unity subordinate to $\{\Gamma_i\}_0^N$ s.t.

$$0 \leq \phi_i \leq 1, \qquad \operatorname{supp} \phi_i \subset \Gamma_i, \qquad \sum_{i=0}^N \phi_i = 1 \text{ in } \overline{\Omega}$$

Hence $f = \sum_{i=0}^N f\phi_i$ in Ω and $\varphi = \sum_{i=1}^N \varphi\phi_i$ on $\partial\Omega$ since $\Gamma_0 \subset\subset \Omega$. By definition of distributional derivative $D(f\phi_0) \in D'$ and that $\operatorname{supp} f\phi_0 \subset \Gamma_0 \subset\subset \Omega$, for any $g \in C_0^1(\mathbb{R}^n; \mathbb{R}^n)$

$$\int_\Omega f \phi_0 \operatorname{div} g \, dx = \int f \phi_0 \operatorname{div} g \, dx = -\int \langle g, D(f\phi_0) \rangle = -\int_\Omega \langle g, D(f\phi_0) \rangle \tag{1.62}$$

while for $i = 1, \cdots, N$, one apply flattening boundary and then (1.47) on each $C_{R_i}^+$ to obtain

$$\int_\Omega f\phi_i \operatorname{div} g \, dx = -\int_\Omega \langle g, D(f\phi_i) \rangle + \int_{\partial\Omega} \varphi\phi_i \langle g, \nu \rangle \, dH_{n-1} \tag{1.63}$$

Hence summing up (1.62) and (1.63) gives (1.61). $\qquad\square$

Proposition 1.2.3 (Approximation in BV implies Approximation in Trace). *$\Omega \subset \mathbb{R}^n$ open and bounded, $\partial\Omega$ Lipschitz. $f \in BV(\Omega)$. If $\{f_j\} \subset BV(\Omega)$ s.t. $f_j \to f$ in $L^1(\Omega)$ and*

$$\lim_{j \to \infty} \int_\Omega |Df_j| = \int_\Omega |Df| \tag{1.64}$$

then, letting φ_j be trace for f_j and φ trace for f

$$\lim_{j \to \infty} \int_{\partial\Omega} |\varphi_j - \varphi| \, dH_{n-1} = 0 \tag{1.65}$$

Remark 1.2.1. *Let $\Omega \subset \mathbb{R}^n$ open and bounded, $\partial\Omega$ Lipschitz. $f \in BV(\Omega)$.*

- *By smooth approximation Theorem 1.1.2, there exists $\{f_j\} \subset C^\infty(\Omega)$ s.t. $\|f_j - f\|_{L^1(\Omega)} \to 0$ and $\lim_{j \to 0} \int_\Omega |Df_j| \, dx = \int_\Omega |Df|$. As in Proposition 1.2.1, or essentially (1.18), letting φ_j be trace for f_j and φ trace for f, one has $\varphi_j = \varphi$ for any j.*

- Let $A \subset\subset \Omega$ open with ∂A Lipschitz. Then $f|_A \in BV(A)$ and $f|_{\Omega \setminus \overline{A}}$, hence denote $f_A^-, f_A^+ \in L^1(\partial A)$ as their trace respectively.

  1. One has immediately via differentiation (1.60) that for H_{n-1}-a.e. $x \in \partial A$

  $$\lim_{\rho \to 0} \frac{1}{\rho^n} \int_{B_\rho(x) \cap A} |f(z) - f_A^-(x)| \, dz = 0 \qquad \lim_{\rho \to 0} \frac{1}{\rho^n} \int_{B_\rho(x) \cap (\Omega \setminus \overline{A})} |f(z) - f_A^+(x)| \, dz = 0 \qquad (1.66)$$

  2. Via Extension property for BV Proposition 1.2.2, denoting ν as outer unit normal w.r.t. ∂A, one has important characterisation for the measures $|Df|$ and Df on ∂A

  $$\int_{\partial A} |Df| = \int_{\partial A} |f_A^+ - f_A^-| \, dH_{n-1}(y) \qquad\qquad (1.67)$$

  $$\int_{\partial A} Df = \int_{\partial A} \left(f_A^+ - f_A^- \right) \nu \, dH_{n-1}(y) \qquad\qquad (1.68)$$

  In particular, let $\Omega = B_R$ and $A = B_\rho$ for $\rho < R$, and denote f_ρ^-, $f_\rho^+ \in L^1(\partial B_\rho)$ as trace for $f|_{B_\rho}$ and $f|_{B_R \setminus \overline{B}_\rho}$ respectively. One has, for some N_1, $N_2 \subset \mathbb{R}$ set measure 0

  $$\lim_{\substack{t \to \rho^- \\ t \notin N_1}} \int_{\partial B_1} |f(tx) - f_\rho^-(\rho x)| \, dH_{n-1}(x) = 0 \qquad \lim_{\substack{t \to \rho^+ \\ t \notin N_2}} \int_{\partial B_1} |f(tx) - f_\rho^+(\rho x)| \, dH_{n-1}(x) = 0 \qquad (1.69)$$

  Proof. It suffices to prove for f_ρ^-. Notice, by a change of variables, for any $\frac{\rho}{2} < t < \rho$

  $$\int_{\partial B_1} |f(tx) - f_\rho^-(\rho x)| \, dH_{n-1}(x) = \frac{1}{\rho^n} \int_{\partial B_\rho} |f(\frac{t}{\rho} x) - f_\rho^-(x)| \, dH_{n-1}(x)$$

  $$\leq \frac{1}{\rho^n} \frac{1}{(\rho - t)^n} \int_{\partial B_\rho} \int_{B_{2(\rho-t)}(x) \cap B_\rho} |f(z) - f_\rho^-(x)| \, dz \, H_{n-1}(x)$$

  where the last inequality holds for a.e. t. Denote the set that it fails by N_1. Now since $f \in L^1(B_R)$, one may apply DCT and use the inner part of (1.66)

  $$\limsup_{\substack{t \to \rho^- \\ t \notin N_1}} \int_{\partial B_1} |f(tx) - f_\rho^-(\rho x)| \, dH_{n-1}(x) \leq \limsup_{\substack{t \to \rho^- \\ t \notin N_1}} \frac{1}{\rho^n} \int_{\partial B_\rho} \frac{1}{(\rho - t)^n} \int_{B_{2(\rho-t)}(x) \cap B_\rho} |f(z) - f_\rho^-(x)| \, dz \, H_{n-1}(x)$$

  $$\leq \frac{1}{\rho^n} \int_{\partial B_\rho} \left(\lim_{\substack{t \to \rho^- \\ t \notin N_1}} \frac{1}{(\rho - t)^n} \int_{B_{2(\rho-t)}(x) \cap B_\rho} |f(z) - f_\rho^-(x)| \, dz \right) H_{n-1}(x)$$

  $$= 0$$

  $\square$

  Also, since $f \in BV(\Omega)$, $|Df|$ is of finite measure. Due to countable additivity of measure for $|Df|$, for a.e. ρ, one has $\int_{\partial B_\rho} |Df| = 0$, hence

  $$f_\rho^+(x) = f(x) = f_\rho^-(x) \qquad for \ H_{n-1} - a.e. \ x \in \partial B_\rho \ for \ a.e. \ \rho \qquad (1.70)$$

- Let $A \subset \Omega$ open with ∂A Lipschitz, and $f \in BV(A)$. One may extend f to Ω by $F := \begin{cases} f & in \ A \\ 0 & in \ \Omega \setminus A \end{cases}$ hence denoting $F_A^-, F_A^+ \in L^1(\partial A)$ as trace for $F|_A$, $F|_{\Omega \setminus \overline{A}}$, one has $F_A^- = f_A^-$ as trace of f on ∂A, and $F_A^+ = 0$.

  1. from (1.67)

  $$\int_\Omega |DF| - \int_A |Df| = \int_{\Omega \cap \partial A} |DF| = \int_{\Omega \cap \partial A} |f_A^-| \, dH_{n-1} \qquad\qquad (1.71)$$

  2. from (1.68), denoting ν as inner unit normal w.r.t. ∂A

  $$\int_\Omega DF - \int_A Df = \int_{\Omega \cap \partial A} DF = \int_{\Omega \cap \partial A} f_A^- \nu \, dH_{n-1} \qquad\qquad (1.72)$$

  In particular, one may further compute 3 perimeters for subsets of Caccioppoli set w.r.t. some ball. Let $\Omega = B_R$ and $A = B_\rho$ for $\rho < R$, and $f = \varphi_E$ for $E \subset \mathbb{R}^n$ Caccioppoli. Then $F = \varphi_{E \cap B_\rho}$. Due to (1.70), for a.e. ρ, $\varphi_E = \varphi_{E,\rho}^-$ for H_{n-1}-a.e. $x \in \partial B_\rho$. Note $\partial B_\rho \cap B_R = \partial B_\rho$, so

1. from (1.71)

$$P(E \cap B_\rho, B_R) = P(E, B_\rho) + H_{n-1}(E \cap \partial B_\rho) \qquad for \ a.e. \ \rho \ s.t. \ (1.70) \ holds \qquad (1.73)$$

2. similarly, from (1.72), denoting ν as inner unit normal w.r.t. ∂B_ρ

$$\int_{B_R} D\varphi_{E \cap B_\rho} = \int_{B_\rho} D\varphi_E + \int_{\partial B_\rho} \varphi_E \, \nu \, dH_{n-1} \qquad for \ a.e. \ \rho \ s.t. \ (1.70) \ holds \qquad (1.74)$$

Now let $A = B_R \setminus \overline{B}_\rho$, then $F = \varphi_{E \cap (B_R \setminus \overline{B}_\rho)}$, so for a.e. ρ, $\varphi_E = \varphi_{E,\rho}^+$ for H_{n-1}-a.e. $x \in \partial B_\rho$

$$P(E \setminus \overline{B}_\rho, B_R) = P(E, B_R \setminus \overline{B}_\rho) + H_{n-1}(E \cap \partial B_\rho) \qquad for \ a.e. \ \rho \ s.t. \ (1.70) \ holds \qquad (1.75)$$

Furthermore for A as above, $B_R \setminus (E \cap (B_R \setminus \overline{B}_\rho)) = (B_R \setminus E) \cap (B_R \setminus \overline{B}_\rho)$, then using that mutual disjoint sets share same perimeter

$$P((B_R \setminus E) \cap (B_R \setminus \overline{B}_\rho), B_R) = P(E \cap (B_R \setminus \overline{B}_\rho), B_R) = P(E \setminus \overline{B}_\rho, B_R)$$

one has, again by mutual disjoint sets sharing same perimeter

$$P(E \cup \overline{B}_\rho, B_R) = P(B_R \setminus (E \cup \overline{B}_\rho), B_R) = P((B_R \setminus E) \cap (B_R \setminus \overline{B}_\rho), B_R) = P(E \setminus \overline{B}_\rho, B_R)$$
$$= P(E, B_R \setminus \overline{B}_\rho) + H_{n-1}(E \cap \partial B_\rho) \qquad for \ a.e. \ \rho \ s.t. \ (1.70) \ holds \qquad (1.76)$$

Hence one may measure perimeter of subsets for E in big ball using perimeter of E in small balls and the boundary quantity $H_{n-1}(E \cap \partial B_\rho)$ via (1.73), (1.75) and (1.76).

1.2.4   Converse to Trace Construction

Theorem 1.2.3 (Converse to Trace Construction). *Let $\varphi \in L^1(\mathscr{B}_R)$ for $R > 0$ and compactly supported. For any $\varepsilon > 0$, there exists $f \in W^{1,1}(C_R^+)$ s.t. φ is trace of f and*

$$\int_{C_R^+} |f| \, dx \leq \varepsilon \int_{\mathscr{B}_R} |\varphi| \, dH_{n-1} \qquad (1.77)$$

$$\int_{C_R^+} |Df| \, dx \leq (1 + \varepsilon) \int_{\mathscr{B}_R} |\varphi| \, dH_{n-1} \qquad (1.78)$$

Proof. There exists $\{\varphi_j\} \subset C^\infty(\mathscr{B}_R)$ s.t. $\|\varphi_j - \varphi\|_{L^1(\mathscr{B}_R)} \to 0$ with $\varphi_0 = 0$, $\|\varphi_j\|_{L^1(\mathscr{B}_R)} \leq 2 \|\varphi\|_{L^1(\mathscr{B}_R)}$ and

$$\int_{\mathscr{B}_R} |\varphi_j - \varphi_{j+1}| \, dH_{n-1} \leq 2^{-j-1} \left(1 + \frac{\varepsilon}{2}\right) \int_{\mathscr{B}_R} |\varphi| \, dH_{n-1} \implies \sum_{j=0}^{\infty} \|\varphi_j - \varphi\|_{L^1(\mathscr{B}_R)} \leq \left(1 + \frac{\varepsilon}{2}\right) \|\varphi\|_{L^1(\mathscr{B}_R)}$$

Now one may construct f with support on neighborhood of $\mathscr{B}_R$. Let $\{t_k\} \subset (0, R)$ be strictly decreasing sequence to 0. Define $f : C_R^+ \to \mathbb{R}$ s.t. for $x \in \mathscr{B}_R$, $t \in (0, R)$

$$f(x, t) := \begin{cases} 0 & if \ t > t_0 \\ \frac{t - t_{k+1}}{t_k - t_{k+1}} \varphi_k(x) + \frac{t_k - t}{t_k - t_{k+1}} \varphi_{k+1}(x) & if \ t_k \geq t > t_{k+1} \ for \ k \geq 0 \end{cases}$$

Hence one may calculate for any $t_k \geq t > t_{k+1}$ for $k \geq 0$

$$|D_i f| \leq |D_i \varphi_k(x)| + |D_i \varphi_{k+1}(x)| \quad 1 \leq i \leq n-1$$
$$|D_n f| \leq \frac{1}{t_k - t_{k+1}} |\varphi_k(x) - \varphi_{k+1}(x)|$$

Hence one calculate $\int_{C_R^+} |f| \, dx$ and $\int_{C_R^+} |Df| \, dx$ s.t.

$$
\begin{aligned}
\int_{C_R^+} |f| \, dx &= \int_0^R \int_{\mathscr{B}_R} |f| \, dH_{n-1}(x) \, dt = \sum_{k=0}^\infty \int_{t_{k+1}}^{t_k} \int_{\mathscr{B}_R} |f| \, dH_{n-1}(x) \, dt \\
&\leq \sum_{k=0}^\infty \int_{t_{k+1}}^{t_k} \left(\|\varphi_k\|_{L^1(\mathscr{B}_R)} + \|\varphi_{k+1}\|_{L^1(\mathscr{B}_R)} \right) dt = \sum_{k=0}^\infty \left(\|\varphi_k\|_{L^1(\mathscr{B}_R)} + \|\varphi_{k+1}\|_{L^1(\mathscr{B}_R)} \right) (t_k - t_{k+1}) \\
&\leq 4 \|\varphi\|_{L^1(\mathscr{B}_R)} \sum_{k=0}^\infty (t_k - t_{k+1}) = 4t_0 \|\varphi\|_{L^1(\mathscr{B}_R)} \\
\int_{C_R^+} |Df| \, dx &= \sum_{k=0}^\infty \int_{t_{k+1}}^{t_k} \int_{\mathscr{B}_R} |Df| \, dH_{n-1}(x) \, dt \leq \sum_{k=0}^\infty \int_{t_{k+1}}^{t_k} \sum_{i=1}^n \int_{\mathscr{B}_R} |D_i f| \, dH_{n-1}(x) \, dt \\
&\leq \sum_{k=0}^\infty \int_{t_{k+1}}^{t_k} \left(\sum_{i=1}^{n-1} \left(\|D_i \varphi_k\|_{L^1(\mathscr{B}_R)} + \|D_i \varphi_{k+1}\|_{L^1(\mathscr{B}_R)} \right) + \frac{1}{t_k - t_{k+1}} \|\varphi_k - \varphi_{k+1}\|_{L^1(\mathscr{B}_R)} \right) dt \\
&\leq \sum_{k=0}^\infty \left(\left(\|D\varphi_k\|_{L^1(\mathscr{B}_R)} + \|D\varphi_{k+1}\|_{L^1(\mathscr{B}_R)} \right) (t_k - t_{k+1}) + \|\varphi_k - \varphi_{k+1}\|_{L^1(\mathscr{B}_R)} \right) \\
&\leq \sum_{k=0}^\infty \left(\|D\varphi_k\|_{L^1(\mathscr{B}_R)} + \|D\varphi_{k+1}\|_{L^1(\mathscr{B}_R)} \right) (t_k - t_{k+1}) + \left(1 + \frac{\varepsilon}{2} \right) \|\varphi\|_{L^1(\mathscr{B}_R)}
\end{aligned}
$$

But one is left to choose t_k freely. Hence choose t_k s.t. $4t_0 < \varepsilon$ and for $k \geq 0$

$$
(t_k - t_{k+1}) \leq \frac{\varepsilon \|\varphi\|_{L^1(\mathscr{B}_R)}}{1 + \|D\varphi_k\|_{L^1(\mathscr{B}_R)} + \|D\varphi_{k+1}\|_{L^1(\mathscr{B}_R)}} 2^{-k-2}
$$

Hence one obtain (1.77) and (1.78), whence $f \in W^{1,1}(C_R^+)$. To see φ really is trace for f, denote $f_t(x) := f(x, t)$ and compute for $t_k \geq t > t_{k+1}$, following construction in Theorem 1.2.1 and DCT

$$
\int_{\mathscr{B}_R} |f_t(x) - \varphi(x)| \, dH_{n-1}(x) \leq \int_{\mathscr{B}_R} \left| \frac{t - t_{k+1}}{t_k - t_{k+1}} \varphi_k(x) - \varphi(x) \right| dH_{n-1}(x) + \int_{\mathscr{B}_R} \left| \frac{t_k - t}{t_k - t_{k+1}} \varphi_{k+1}(x) - \varphi(x) \right| dH_{n-1}(x) \overset{k\to\infty}{\to} 0
$$

Hence by uniqueness of L^1 limits, φ is indeed trace for f.

$\square$

Theorem 1.2.4 (Converse to Trace Construction). $\Omega \subset \mathbb{R}^n$ *open bounded,* $\partial\Omega$ *Lipschitz.* $\varphi \in L^1(\partial\Omega)$. *Then for any* $\varepsilon > 0$, *there exists* $f \in W^{1,1}(\Omega)$ *s.t.* φ *is trace of* f *and*

$$
\int_\Omega |f| \, dx \leq \varepsilon \int_{\partial\Omega} |\varphi| \, dH_{n-1} \tag{1.79}
$$

$$
\int_\Omega |Df| \, dx \leq A \int_{\partial\Omega} |\varphi| \, dH_{n-1} \tag{1.80}
$$

for $A = A(\partial\Omega)$ *but independent of* f, φ, ε. *If moreover* $\partial\Omega$ *is* C^1, *one may choose* $A = (1 + \varepsilon)$. *Also,* f *may be taken to be supported on arbitrary small neighborhood of* $\partial\Omega$ *by controlling* t_0 *via* ε.

Chapter 2

Reduced Boundary

2.1 Construction and Properties

As a preliminary, one finds substitution for general Borel sets so that their measure theoretic boundary and topological boundary agree. We work with sets satisfying Lemma 2.1.1 from later on.

Lemma 2.1.1. *Let $E \subset \mathbb{R}^n$ Borel. Then there exists $\tilde{E}$ Borel s.t. $|\tilde{E}\Delta E| = 0$ differ by Lebesgue measure 0 and*

$$0 < |\tilde{E} \cap B_\rho(x)| < \omega_n \rho^n \quad \text{for any } \rho > 0 \text{ and } x \in \partial \tilde{E} \tag{2.1}$$

Proof. Define

$$E_0 := \{x \in \mathbb{R}^n \mid \text{ there exists } \rho > 0 \text{ s.t. } |E \cap B_\rho(x)| = 0\}$$
$$E_1 := \{x \in \mathbb{R}^n \mid \text{ there exists } \rho > 0 \text{ s.t. } |E \cap B_\rho(x)| = |B_\rho(x)| = \omega_n \rho^n\}$$

One see both E_0 and E_1 are open. For $x \in E_0$, take $\rho > 0$ s.t. $|E \cap B_\rho(x)| = 0$. Then for any $y \in B_\rho(x)$, let $\rho_0 := \rho - |x - y|$, so $B_{\rho_0}(y) \subset B_\rho(x)$ hence $|E \cap B_{\rho_0}(y)| = 0$. Due to existence of ρ_0, $y \in E_0$, i.e., the neighborhood $B_\rho(x) \subset E_0$. So E_0 open. For $x \in E_1$, there exists $\rho > 0$ s.t. $|E \cap B_\rho(x)| = |B_\rho(x)|$, i.e., $|B_\rho(x) \cap E^c| = 0$. Again, for any $y \in B_\rho(x)$, let $\rho_0 := \rho - |x - y|$, so $B_{\rho_0}(y) \subset B_\rho(x)$, thus $|B_{\rho_0}(y) \cap E^c| = 0$. Hence $y \in E_1$, we have $B_\rho(x) \subset E_1$, so E_1 is open. One may further show that $|E_0 \cap E| = 0$. Since for any $x \in E_0$, one may choose ρ_x s.t. $|E \cap B_{\rho_x}(x)| = 0$, and it indeed covers $E_0 \subset \bigcup_{x \in E_0} B_{\rho_x}(x)$, we may choose sequence $\{x_j\} \subset E_0$ as index for covering. One compute, due to $|B_{\rho_{x_j}}(x_j) \cap E| = 0$ for any j

$$|E_0 \cap E| \le |\bigcup_{j=1}^\infty B_{\rho_{x_j}}(x_j) \cap E| \le \sum_{j=1}^\infty |B_{\rho_{x_j}}(x_j) \cap E| = 0$$

Similarly, $|E_1 \setminus E| = 0$ by replacing E in above computation with E^c. Since E_0, E_1 open, $\tilde{E} := (E \cup E_1) \setminus E_0$ is Borel. And indeed one has $|\tilde{E}\Delta E| = 0$ via the following

$$|E \setminus \tilde{E}| = |E \cap ((E \cup E_1) \setminus E_0)^c| = |E \cap ((E \cup E_1)^c \cup E_0)| = |(E \cap E^c \cap E_1^c) \cup (E \cap E_0)| = |E_0 \cap E| = 0$$
$$|\tilde{E} \setminus E| = |(E \cup E_1) \cap E_0^c \cap E^c| = |(E \cap E_0^c \cap E^c) \cup (E_1 \cap E_0^c \cap E^c)| \le |E_1 \setminus E| = 0$$

Now for any $x \in \partial \tilde{E}$, since E_0, E_1 open, $x \notin E_0 \cup E_1$. Hence for any $\rho > 0$, (2.1) holds. $\square$

2.1.1 Reduced Boundary and Uniform Density Estimate

Definition 2.1.1 (Reduced Boundary). *Given $E \subset \mathbb{R}^n$ Caccioppoli. $x \in \partial^* E$ reduced boundary if*

$$\int_{B_\rho(x)} |D\varphi_E| > 0 \quad \text{for any } \rho > 0 \tag{2.2}$$

and hence, defining

$$\nu_\rho(x) := \frac{\int_{B_\rho(x)} D\varphi_E}{\int_{B_\rho(x)} |D\varphi_E|} \quad \text{for any } \rho > 0 \tag{2.3}$$

One require the limits $\lim_{\rho \to 0} \nu(x)$ exists and has length 1

$$\nu(x) := \lim_{\rho \to 0} \nu(x) = \lim_{\rho \to 0} \frac{\int_{B_\rho(x)} D\varphi_E}{\int_{B_\rho(x)} |D\varphi_E|} \tag{2.4}$$

$$|\nu(x)| = 1 \tag{2.5}$$

i.e., $\partial^* E := \{x \in \partial E \mid (2.2) \text{ holds for any } \rho > 0, \text{ and the limiting object } (2.3) \text{ satisfies } (2.4) \text{ and } (2.5)\}$

Recall the Lebesgue-Besicovitch differentiation.

Lemma 2.1.2 (Lebesgue-Besicovitch differentiation). μ_1, μ_2 *Borel measures on* $\mathbb{R}^n$, *then*

$$D_{\mu_2}\mu_1 := \lim_{\rho \to 0} \frac{\mu_1(B_\rho(x))}{\mu_2(B_\rho(x))}$$

is defined μ_2*-a.e. on* $\mathbb{R}^n$, *and* $D_{\mu_2}\mu_1 \in L^1_{loc}(\mathbb{R}^n, \mu_2)$. *If furthermore,* $\mu_1 \ll \mu_2$, *i.e.,* μ_1 *is absolutely continuous w.r.t.* μ_2 *in the sense that* $\mu_2(E) = 0$ *implies* $\mu_1(E)$ *for any* $E \subset \mathbb{R}^n$ *Borel, then we write*

$$\mu_1 = D_{\mu_2}\mu_1 \cdot \mu_2 \qquad \text{on all Borel sets}$$

Remark 2.1.1. *Note* $D\varphi_E$ *is indeed absolutely continuous w.r.t.* $|D\varphi_E|$. *Hence apply Lemma 2.1.2, one has*

$$\nu(x) := \lim_{\rho \to 0} \frac{\int_{B_\rho(x)} D\varphi_E}{\int_{B_\rho(x)} |D\varphi_E|} \text{ exists and } |\nu(x)| = 1 \qquad |D\varphi_E| - a.e.\ x \in \mathbb{R}^n \tag{2.6}$$

and the following measures agree

$$D\varphi_E = \nu|D\varphi_E| \qquad \text{on all Borel sets} \tag{2.7}$$

In particular, the set $\partial E \setminus \partial^* E$ *has* $|D\varphi_E|$*-measure zero.*

Example 2.1.1. *One has 2 examples. One for smooth boundary and one for Lipschitz.*

- *Let* $E \subset \mathbb{R}^n$ *be bounded, Caccioppoli with* C^2 *boundary* ∂E. *Then* $\partial^* E = \partial E$.

  Proof. Let $A = E$ and $f = \varphi_E$ in (1.68), one has via Extension property for $\varphi_E \in BV(\mathbb{R}^n)$ that

  $$D\varphi_E = \nu dH_{n-1} \quad \text{on } \partial E$$

  where ν denote **inner unit normal w.r.t.** ∂E. And because supp$D\varphi_E \subset \partial E$, one writes for any $\rho > 0$

  $$\int_{B_\rho(x)} D\varphi_E = \int_{B_\rho(x) \cap \partial E} \nu\, dH_{n-1}$$

  while C^2 boundary ensure via (1.4) that

  $$\int_{B_\rho(x)} |D\varphi_E| = H_{n-1}(B_\rho(x) \cap \partial E)$$

  hence one has explicit formula for ν_ρ

  $$\nu_\rho(x) = \frac{\int_{B_\rho(x) \cap \partial E} \nu\, dH_{n-1}}{H_{n-1}(B_\rho(x) \cap \partial E)} \qquad for\ any\ x \in \partial E$$

  Since $\nu \in C^1(\partial E; \mathbb{R}^n)$, differentiation gives $\lim_{\rho \to 0} \nu_\rho(x) = \nu(x)$ for any $x \in \partial E$. Hence $|\nu| = 1$ as inherited.
  $\square$

- *Let* $E = (0,1) \times (0,1) \subset \mathbb{R}^2$. *Notice except for the four corners, the boundaries are piecewise* C^∞, *hence these parts belong to* $\partial^* E$. *Now for any corner* x, *one may compute*

  $$|\nu(x)| = \lim_{\rho \to 0} \frac{|\int_{B_\rho(x)} D\varphi_E|}{\int_{B_\rho(x)} |D\varphi_E|} = \frac{1}{\sqrt{2}}$$

  Hence the four corners do not belong to $\partial^* E$.

One has Uniform Density estimates, which says bounded oscillation in normal directions at a given boundary point $x \in \partial E$ prevents densities of E and E^c from disappearing under blow-up limit. In particular, if $x \in \partial^* E$, it indeed satisfies our assumption, so uniform density estimate holds. For simplicity, let $0 \in \partial E$ via translation.

Theorem 2.1.1 (Uniform Density Estimates). $E \subset \mathbb{R}^n$ *be Caccioppoli and* $0 \in \partial E$. *If there exists* $\rho_0 > 0$ *and* $q > 0$ *constants s.t. for any* $\rho < \rho_0$

$$\int_{B_\rho} |D\varphi_E| > 0$$

$$|\nu_\rho(0)| = \left| \frac{\int_{B_\rho} D\varphi_E}{\int_{B_\rho} |D\varphi_E|} \right| \geq q > 0 \tag{2.8}$$

Then for any $\rho < \rho_0$, one has uniform estimates on the density

$$\frac{|E \cap B_\rho|}{\rho^n} \geq C_1(n, q) > 0 \tag{2.9}$$

$$\frac{|E^c \cap B_\rho|}{\rho^n} \geq C_2(n, q) > 0 \tag{2.10}$$

$$0 < C_3(n, q) \leq \frac{\int_{B_\rho} |D\varphi_E|}{\rho^{n-1}} \leq C_4(n, q) < \infty \tag{2.11}$$

for constants C_1, C_2, C_3, C_4 only relevant to n, q.

Proof. Since E Caccioppoli, $\varphi_E \in BV(B_{\rho_0})$. Denoting ν as inner unit normal w.r.t. ∂B_ρ one has via (1.74)

$$\int D\varphi_{E \cap B_\rho} = \int_{B_\rho} D\varphi_E + \int_{\partial B_\rho} \varphi_E \nu \, dH_{n-1} \qquad for \ a.e. \ \rho < \rho_0$$

evaluate the vector-valued measure on some constant unit vector $e \in \mathbb{S}^{n-1}$ gives, for ρ s.t. (1.74) holds

$$0 = -\int \operatorname{div}(e) \, \varphi_{E \cap B_\rho} = \int \langle e, D\varphi_{E \cap B_\rho} \rangle = \int_{B_\rho} \langle e, D\varphi_E \rangle + \int_{\partial B_\rho} \varphi_E \nu \cdot e \, dH_{n-1}$$

Hence for any $e \in \mathbb{S}^{n-1}$

$$\left| \int_{B_\rho} \langle e, D\varphi_E \rangle \right| = \left| \int_{\partial B_\rho} \varphi_E \nu \cdot e \, dH_{n-1} \right| \leq \int_{\partial B_\rho} \varphi_E \, dH_{n-1} = H_{n-1}(E \cap \partial B_\rho) \leq C\rho^{n-1}$$

taking supremum on LHS and using Riesz Representation yields

$$\left| \int_{B_\rho} D\varphi_E \right| \leq H_{n-1}(E \cap \partial B_\rho) \tag{2.12}$$

Using (2.12) and (2.8) further gives

$$\int_{B_\rho} |D\varphi_E| \leq \frac{1}{q} \left| \int_{B_\rho} D\varphi_E \right| \leq C_4 \rho^{n-1} \quad for \ a.e. \ \rho < \rho_0 \ s.t. \ (1.74) \ holds$$

Now using continuity from above of the measure $|D\varphi_E|$, we conclude the second part to (2.11) for all $\rho < \rho_0$. Now, using (1.73) and similar reasons as above, one has

$$P(E \cap B_\rho) = P(E, B_\rho) + H_{n-1}(E \cap \partial B_\rho) \quad for \ a.e. \ \rho < \rho_0$$

$$= \int_{B_\rho} |D\varphi_E| + \int_{\partial B_\rho} \varphi_E \, dH_{n-1} \leq \left(\frac{1}{q} + 1 \right) \int_{\partial B_\rho} \varphi_E \, dH_{n-1}$$

Since $E \cap B_\rho$ is bounded Caccioppoli, via isoperimetric inequality (1.40) and noting $P(E \cap B_\rho) = \int |D\varphi_{E \cap B_\rho}|$

$$|E \cap B_\rho|^{\frac{n-1}{n}} \leq \left(\frac{1}{q} + 1 \right) C(n) \int_{\partial B_\rho} \varphi_E \, dH_{n-1} \tag{2.13}$$

for some $C(n)$ from (1.40). Notice by coarea formula, denoting $g(\rho) = |E \cap B_\rho|$

$$g(R) = |E \cap B_R| = \int_{B_R} \varphi_E \, dx = \int_0^R \int_{\partial B_\rho} \varphi_E \, dH_{n-1} \, d\rho \implies g'(\rho) = \int_{\partial B_\rho} \varphi_E \, dH_{n-1}$$

Hence (2.13) writes

$$g(\rho)^{\frac{n-1}{n}} \leq \left(\frac{1}{q} + 1 \right) C(n) g'(\rho) \implies \rho \leq \left(\frac{1}{q} + 1 \right) C(n) \, n \, g(\rho)^{\frac{1}{n}} \implies \left(\frac{1}{C(n) \, n \left(\frac{1}{q} + 1 \right)} \right)^n \leq \frac{|E \cap B_\rho|}{\rho^n}$$

denoting $C_1 := \left(\frac{1}{C(n) n (\frac{1}{q} + 1)} \right)^n$ and using continuity from below of the measure $|E \cap B_\rho|$ in ρ, one conclude (2.9) for every $\rho < \rho_0$. Note for E^c, $D\varphi_{E^c} = -D\varphi_E$ due to for any $g \in C_0^1(\mathbb{R}^n; \mathbb{R}^n)$

$$\int \langle g, D\varphi_{E^c} \rangle = -\int \varphi_{E^c} \operatorname{div}(g) \, dx = -\int (1 - \varphi_E) \operatorname{div}(g) \, dx = \int \varphi_E \operatorname{div}(g) \, dx = -\int \langle g, D\varphi_E \rangle$$

whence $|D\varphi_E| = |D\varphi_{E^c}|$ and the above same argument runs with $C_2 = C_1$, resulting in (2.10). To see first part to (2.11), notice from (2.9) and (2.10), one has

$$C_1 \rho^n \leq \min\{|E \cap B_\rho|, |E^c \cap B_\rho|\} \implies C_1^{\frac{n-1}{n}} \rho^{n-1} \leq \min\{|E \cap B_\rho|, |E^c \cap B_\rho|\}^{\frac{n-1}{n}}$$

Hence applying Poincaré inequality (1.41) one has, for some $\tilde{C}(n) > 0$

$$C_1^{\frac{n-1}{n}} \rho^{n-1} \leq \tilde{C}(n) \int_{B_\rho} |D\varphi_E| \implies 0 < \frac{C_1^{\frac{n-1}{n}}}{\tilde{C}(n)} \leq \frac{1}{\rho^{n-1}} \int_{B_\rho} |D\varphi_E|$$

define $C_3 := \frac{C_1^{\frac{n-1}{n}}}{\tilde{C}(n)}$ yields the first part of (2.11). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.1.2   Blow-up Limit

One define the tangent plane and half spaces for given $z \in \partial^* E$ (hence $\nu(z)$ is well-defined and $|\nu(z)| = 1$)

- Tanget Hyperplane to $\partial^* E$ at z is $T(z) := \{x \in \mathbb{R}^n \mid \langle \nu(z), x - z \rangle = 0\}$

- Half spaces to $\partial^* E$ at z on the same and opposite side with $\nu(z)$ are respectively

$$T^+(z) := \{x \in \mathbb{R}^n \mid \langle \nu(z), x - z \rangle > 0\}$$
$$T^-(z) := \{x \in \mathbb{R}^n \mid \langle \nu(z), x - z \rangle < 0\}$$

One may now show that the blowup limit of a point in reduced boundary actually converges to the half space on the same side as the outer normal. For simplicity, via translation and rotation, one assume $0 \in \partial^* E$, and the inner normal $\nu(0)$ is parallel to the x_1-axis that points towards $-\infty$. One wish to obtain the limit $T^+(0)$. But before the proof, one needs a De La Vallée Poussin Theorem to guarantee convergence in LHS of (2.16) given the L^1_{loc} convergence.

Lemma 2.1.3 (De La Vallée Poussin Theorem). *Given E_j sequence of Caccioppoli Sets in $\mathbb{R}^n$. Suppose $\varphi_{E_j} \to \varphi_E$ in $L^1_{loc}(\mathbb{R}^n)$ and that $\int_{\mathbb{R}^n} |D\varphi_{E_j}| \leq M < \infty$ the total variation is uniformly bounded. Then up to a subsequence, the convergence holds in vague topology*

$$\int g \cdot D\varphi_{E_j} \to \int g \cdot D\varphi_E \quad \forall \, g \in C_0^1(\mathbb{R}^n)$$

and for a.e. ρ

$$\lim_{j \to \infty} \int_{B_\rho} D\varphi_{E_j} = \int_{B_\rho} D\varphi_E \tag{2.14}$$

Theorem 2.1.2 (Blow-up Limit of Reduced Boundary). *$E \subset \mathbb{R}^n$ Caccioppoli. $0 \in \partial^* E$ with $\nu(0) = (-1, 0, \cdots, 0)$. For any $t > 0$, define the set for blowup*

$$E_t := \{x \in \mathbb{R}^n \mid t \, x \in E\} \tag{2.15}$$

Then there exists a subsequence $t_j \to 0^+$ s.t. $E_j := E_{t_j} \to T^+ := T^+(0)$ in $L^1_{loc}(\mathbb{R}^n)$ sense. Moreover, for every open set $A \subset \mathbb{R}^n$ s.t. $H_{n-1}(\partial A \cap T(0)) = 0$

$$\lim_{t_j \to 0} \int_A |D\varphi_{E_j}| = \int_A |D\varphi_{T^+}| = H_{n-1}(T(0) \cap A) \tag{2.16}$$

Proof. One wish to extract a convergent subsequence using compactness argument. First note in our setting, the targeting limit is $T^+ = \{x \in \mathbb{R}^n \mid x_1 < 0\}$. Fix $\rho > 0$. Now by change of variables, for any $g \in C_0^1(B_\rho; \mathbb{R}^n)$, write $\tilde{g}(x) := g(x/t)$

$$\int_{B_\rho} \langle g, D\varphi_{E_t} \rangle = -\int_{B_\rho} \operatorname{div}(g(x)) \, \varphi_{E_t}(x) \, dx = -\int_{B_\rho} \operatorname{div}(\tilde{g}(tx)) \, \varphi_E(tx) \, dx$$

$$= -\int_{B_\rho} t \operatorname{div}(\tilde{g})(tx) \, \varphi_E(tx) \, dx = -\frac{1}{t^{n-1}} \int_{B_{t\rho}} \operatorname{div}(\tilde{g})(y) \, \varphi_E(y) \, dy$$

$$= \frac{1}{t^{n-1}} \int_{B_{t\rho}} \langle \tilde{g}, D\varphi_E \rangle \implies \int_{B_\rho} D\varphi_{E_t} = \frac{1}{t^{n-1}} \int_{B_{t\rho}} D\varphi_E \tag{2.17}$$

And by considering total variation, one has

$$\int_{B_\rho} |D\varphi_{E_t}| = \frac{1}{t^{n-1}} \int_{B_{t\rho}} |D\varphi_E| \tag{2.18}$$

With tools (2.17) and (2.18), one proceeds in two directions. First, making use of $0 \in \partial^* E$, in particular (2.4)

$$\lim_{t \to 0} \frac{1}{\int_{B_\rho} |D\varphi_{E_t}|} \begin{pmatrix} \int_{B_\rho} D_1 \varphi_{E_t} \\ \int_{B_\rho} D_2 \varphi_{E_t} \\ \vdots \\ \int_{B_\rho} D_n \varphi_{E_t} \end{pmatrix} = \lim_{t \to 0} \frac{\int_{B_\rho} D\varphi_{E_t}}{\int_{B_\rho} |D\varphi_{E_t}|} = \lim_{t \to 0} \frac{\int_{B_{t\rho}} D\varphi_E}{\int_{B_{t\rho}} |D\varphi_E|} = \nu(0) = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (2.19)$$

Second, one make an immediate observation that for each $\rho > 0$, $\{\varphi_{E_t}\}_t \subset BV(B_\rho)$ because E is Caccioppoli, and for each t, $B_{t\rho}$ is bounded, hence $\varphi_E \in BV(B_{t\rho})$ and RHS of (2.18) is bounded. An immediate consequence is that E_t are Caccioppoli Set for any t. Again, since $0 \in \partial^* E$, one has uniform density estimate. Applying second part of (2.11), together with (2.18) yields

$$\limsup_{t \to 0} \int_{B_\rho} |D\varphi_{E_t}| = \limsup_{t \to 0} \frac{1}{t^{n-1}} \int_{B_{t\rho}} |D\varphi_E| \le C < \infty \quad (2.20)$$

Hence the sequence of functions $\{\varphi_{E_t}\}$ is uniformly bounded in $BV(B_\rho)$ norm for each $\rho > 0$. Thus by compactness theorem 1.1.4, there exists a subsequence $\{\varphi_{E_j}\}$ where $E_j := E_{t_j}$ s.t. $\varphi_{E_j} \to f$ in $L^1_{loc}(\mathbb{R}^n)$ (by unique limit on each ball B_ρ) and that $f \in BV(\mathbb{R}^n)$. Since f is L^1 limit of characteristic functions, $f = \varphi_C$ for some Borel set $C \subset \mathbb{R}^n$. Since $\varphi_C \in BV(\mathbb{R}^n)$, indeed C is Caccioppoli. Moreover, by De La Vallée Poussin Theorem (2.14), for a.e. ρ s.t. $\int_{\partial B_\rho} |D\varphi_C| = 0$, one has approximation in vector-valued radon measure

$$\lim_{t_j \to 0} \int_{B_\rho} D\varphi_{E_j} = \int_{B_\rho} D\varphi_C \quad (2.21)$$

hence combining with (2.19) gives, for the x_1 direction

$$\lim_{t_j \to 0} \int_{B_\rho} |D\varphi_{E_j}| = -\lim_{t_j \to 0} \int_{B_\rho} D_1 \varphi_{E_j} = -\int_{B_\rho} D_1 \varphi_C$$

Now since $\varphi_{E_j} \to \varphi_C$ in $L^1_{loc}(\mathbb{R}^n)$, by semicontinuity 1.1.1

$$\int_{B_\rho} |D\varphi_C| \le \lim_{t_j \to 0} \int_{B_\rho} |D\varphi_{E_j}| = -\int_{B_\rho} D_1 \varphi_C \quad (2.22)$$

but since any other $\int_{B_\rho} D_i \varphi_C = 0$ for $i \ge 2$ as in (2.19), the equality in (2.22) holds. Now by Lebesgue-Besicovitch Differentiation 2.1.2

$$D_1 \varphi_C = \left(\lim_{t \to 0} \frac{\int_{B_\rho} D_1 \varphi_C}{\int_{B_\rho} |D\varphi_C|} \right) |D\varphi_C| = -|D\varphi_C| \qquad \text{on all Borel sets}$$

$$D\varphi_C = \left(\lim_{t \to 0} \frac{\int_{B_\rho} D\varphi_C}{\int_{B_\rho} |D\varphi_C|} \right) |D\varphi_C| = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} |D\varphi_C| \qquad \text{on all Borel sets}$$

Hence $D_i \varphi_C = 0$ as Borel measure for $i \ge 2$. Therefore φ_C depends only on x_1 and $D_1 \varphi_C < 0$ implies φ_C is non-increasing in x_1. Thus $C = \{x \in \mathbb{R}^n \mid x_1 < \lambda\}$ a.e. for some $\lambda \in \mathbb{R}$. One wish to determine λ. Suppose $\lambda < 0$, then we may construct ball $B_{|\lambda|}$ around 0 that does not intersect C, so using $\varphi_{E_j} \to \varphi_C$ in $L^1_{loc}(\mathbb{R}^n)$

$$0 = |C \cap B_{|\lambda|}| = \int_{B_{|\lambda|}} \varphi_C(x) \, dx = \lim_{t_j \to 0} \int_{B_{|\lambda|}} \varphi_{E_j}(x) \, dx$$

$$= \lim_{t_j \to 0} \frac{1}{t_j^n} \int_{B_{|\lambda|}} \varphi_E(t_j x) \, d(t_j x) = \lim_{t_j \to 0} \frac{1}{t_j^n} \int_{B_{|\lambda| t_j}} \varphi_E(y) \, dy$$

$$= \lim_{t_j \to 0} \frac{|E \cap B_{|\lambda| t_j}|}{t_j^n} \ge C_1 > 0$$

for some C_1 from (2.9), contradicting our assumption. If $\lambda > 0$, use

$$0 = |C^c \cap B_{|\lambda|}| = \int_{B_{|\lambda|}} \varphi_{C^c}(x) \, dx = \lim_{t_j \to 0} \int_{B_{|\lambda|}} \varphi_{E_j^c}(x) \, dx$$

$$= \lim_{t_j \to 0} \frac{1}{t_j^n} \int_{B_{|\lambda| t_j}} \varphi_{E^c}(y) \, dy = \lim_{t_j \to 0} \frac{|E^c \cap B_{|\lambda| t_j}|}{t_j^n} \ge C_2 > 0$$

for some C_2 from (2.10). Hence $\lambda = 0$, and so $C = T^+ = \{x \in \mathbb{R}^n \mid x_1 < 0\}$ a.e. It remains to show for any open set $A \subset \mathbb{R}^n$ s.t. $H_{n-1}(\partial A \cap T(0)) = 0$, (2.16) holds. First note that, since T^+ has smooth boundary, one use remark 1.1.1 so that $|D\varphi_{T^+}| = H_{n-1} \llcorner \partial T^+ = H_{n-1} \llcorner T(0)$ as Borel measures. So if $H_{n-1}(\partial A \cap T(0)) = 0$ for some A open, in fact $\int_{\partial A} |D\varphi_{T^+}| = 0$. But this is condition for (1.8) where the equality in semicontinuity holds in subdomains. Hence apply (1.8), one directly arrives at (2.16).

$\square$

Corollary 2.1.1 (Density Estimates on single side of Tangent Plane to Reduced Boundary). *Let $E \subset \mathbb{R}^n$ Caccioppoli, and $0 \in \partial^* E$ with $\nu(0) = (-1, 0, \cdots, 0)$. Then the volumne density on single side vanishes*

$$\lim_{\rho \to 0} \frac{1}{\rho^n} |E \cap B_\rho \cap T^-| = 0 \tag{2.23}$$

$$\lim_{\rho \to 0} \frac{1}{\rho^n} |(B_\rho \setminus E) \cap T^+| = 0 \tag{2.24}$$

and for any $\rho, \varepsilon > 0$, denoting

$$S_{\rho, \varepsilon} := B_\rho \cap \{x \in \mathbb{R}^n \mid |\langle \nu(0), x \rangle| < \varepsilon\rho\} = B_\rho \cap \{x \in \mathbb{R}^n \mid |x_1| < \varepsilon\rho\}$$

the perimeter density takes up constant portion for any $\varepsilon > 0$

$$\lim_{\rho \to 0} \frac{1}{\rho^{n-1}} \int_{S_{\rho, \varepsilon}} |D\varphi_E| = \omega_{n-1} \tag{2.25}$$

where ω_{n-1} is volumne of $n-1$-dim unit ball.

Proof. Under definition (2.15), $T_\rho^+ = T^+$ and $T_\rho^- = T^-$ for any $\rho > 0$. By change of variables as in (2.17)

$$\frac{1}{\rho^n} |E \cap B_\rho \cap T^-| = \frac{1}{\rho^n} \int_{B_\rho} \varphi_E(x) \varphi_{T^-}(x) \, dx = \int_{B_1} \varphi_E(\rho y) \, \varphi_{T^-}(\rho y) \, dy$$

$$= \int_{B_1} \varphi_{E_\rho}(y) \, \varphi_{T_\rho^-}(y) \, dy = |E_\rho \cap B_1 \cap T^-|$$

$$\frac{1}{\rho^n} |(B_\rho \setminus E) \cap T^+| = \frac{1}{\rho^n} \int_{B_\rho} \varphi_{E^c}(x) \, \varphi_{T^+}(x) \, dx = \int_{B_1} \varphi_{E^c}(\rho y) \, \varphi_{T^+}(\rho y) \, dy$$

$$= \int_{B_1} \varphi_{E_\rho^c}(y) \, \varphi_{T_\rho^+}(y) \, dy = |(B_1 \setminus E_\rho) \cap T^+|$$

But from Theorem 2.1.2, $E_\rho \to T^+$ in $L^1_{loc}(\mathbb{R}^n)$ up to a subsequence, hence

$$\lim_{\rho \to 0} \frac{1}{\rho^n} |E \cap B_\rho \cap T^-| = \lim_{\rho \to 0} |E_\rho \cap B_1 \cap T^-| = |T^+ \cap B_1 \cap T^-| = 0$$

$$\lim_{\rho \to 0} \frac{1}{\rho^n} |(B_\rho \setminus E) \cap T^+| = \lim_{\rho \to 0} |(B_1 \setminus E_\rho) \cap T^+| = |(B_1 \setminus T^+) \cap T^+| = 0$$

so (2.23) and (2.24) hold. Moreover, by the exact same procedure with $S_{\rho, \varepsilon}$ in place of B_ρ and $S_{1, \varepsilon}$ in place of B_1 as in (2.17), one has

$$\frac{1}{\rho^{n-1}} \int_{S_{\rho, \varepsilon}} |D\varphi_E| = \int_{S_{1, \varepsilon}} |D\varphi_{E_\rho}|$$

and since $S_{1, \varepsilon}$ is open set with $H_{n-1}(\partial S_{1, \varepsilon} \cap T) = 0$, apply (2.16) to conclude (2.25)

$$\lim_{\rho \to 0} \frac{1}{\rho^{n-1}} \int_{S_{\rho, \varepsilon}} |D\varphi_E| = \lim_{\rho \to 0} \int_{S_{1, \varepsilon}} |D\varphi_{E_\rho}| = H_{n-1}(T \cap S_{1, \varepsilon}) = \omega_{n-1}$$

$\square$

The above Corollary 2.1.1 says for small enough balls $B_\rho(x)$, most of $E \cap B_\rho(x)$ lies in T^+, the same side w.r.t. inner normal ν; while most of $B_\rho(x) \setminus E$ lies in T^-, the outside. For small enough balls, the hyperplane T splits B_ρ into 2 parts which nearly corresponds to inner part E and outside part $\mathbb{R}^n \setminus E$.

2.2 Regularity of Reduced Boundary

The purpose of this section is to argue that for $E \subset \mathbb{R}^n$ Caccioppoli

- $\partial^* E$ is countable union of C^1 hypersurfaces up to set of $|D\varphi_E|$-measure zero.

- $\int_\Omega |D\varphi_E| = H_{n-1}(\partial^* E \cap \Omega)$ so $|D\varphi_E| = H_{n-1} \llcorner \partial^* E$ as Radon measures.

- $\partial^* E$ is dense in ∂E.

One shall first recall the precise definition for Hausdorff measure.

Definition 2.2.1. *Let $A \subset \mathbb{R}^n$, $0 \le k < \infty$ and $0 < \delta \le \infty$. We define the k-dim Hausdorff outer measure at step δ*

$$H_k^\delta(A) := \frac{\omega_k}{2^k} \inf \left\{ \sum_{j=1}^\infty \operatorname{diam}(S_j)^k \mid A \subset \bigcup_{j=1}^\infty S_j, \ \operatorname{diam}(S_j) < \delta \ \forall j \right\} \tag{2.26}$$

and consequently define

$$H_k(A) := \lim_{\delta \to 0} H_k^\delta(A) = \sup_{0 < \delta \le \infty} H_k^\delta(A)$$

as k-dim Hausdorff measure. Here $\omega_k := \Gamma(\frac{1}{2})^k / \Gamma(\frac{k}{2}+1)$ for $k \ge 0$ is measure of unit ball in $\mathbb{R}^k$.

Lemma 2.2.1 (Ratio Estimate). *$E \subset \mathbb{R}^n$ Caccioppoli. $B \subset \partial^* E$. Then*

$$H_{n-1}(B) \le 2 \cdot 3^{n-1} \int_B |D\varphi_E| \tag{2.27}$$

Proof. Since $|D\varphi_E|$ is Radon measure on $\mathbb{R}^n$, it can be approximated from the outside by open sets. Given B, for any $\eta > 0$, there exists $B \subset A$ open s.t.

$$\int_A |D\varphi_E| \le \int_B |D\varphi_E| + \eta \tag{2.28}$$

Moreover, for any $\varepsilon > 0$, apply (2.25) to arbitrary $x \in B$, there exists $0 < \rho(x) < \varepsilon$ s.t.

$$B_{\rho(x)}(x) \subset A \qquad and \qquad \int_{B_{\rho(x)}(x)} |D\varphi_E| \ge \frac{1}{2}\rho(x)^{n-1}\omega_{n-1} \tag{2.29}$$

One think about covering B using balls $\{B_{\rho(x)}(x)\}$ via lemma 1.2.3. So there exists $\{x_i\} \subset B$ s.t.

$$B \subset \bigcup_{i=1}^\infty B_{3\rho(x_i)}(x_i) \qquad and \qquad B_{\rho(x_i)}(x_i) \cap B_{\rho(x_j)}(x_j) = \varnothing \ for \ i \ne j$$

and (2.29) holds for each x_i. Hence one may bound, using $B_{\rho(x_i)}(x_i) \subset A$ and disjoint, and then (2.28)

$$\sum_{i=1}^\infty (3\rho(x_i))^{n-1} \le \sum_{i=1}^\infty 3^{n-1} \frac{2}{\omega^{n-1}} \int_{B_{\rho(x_i)}(x_i)} |D\varphi_E| \le \frac{2 \cdot 3^{n-1}}{\omega^{n-1}} \int_A |D\varphi_E|$$

$$\le \frac{2 \cdot 3^{n-1}}{\omega^{n-1}} \left(\int_B |D\varphi_E| + \eta \right)$$

Hence recalling (2.26), since $B \subset \bigcup_{i=1}^\infty B_{3\rho(x_i)}(x_i)$ with $\rho(x_i) < \varepsilon$ universal bound in i

$$H_{n-1}(B) \le \lim_{\varepsilon \to 0} \frac{\omega_{n-1}}{2^{n-1}} \inf \left\{ \sum_{i=1}^\infty (2 \cdot 3\rho(x_i))^{n-1} \mid \rho(x_i) < \varepsilon \right\} \le 2 \cdot 3^{n-1} \left(\int_B |D\varphi_E| + \eta \right)$$

take $\eta \to 0$ to conclude (2.27). $\qquad\square$

One shall be precise of our notion of C^1 hypersurface.

Definition 2.2.2. *Let Γ_{n-1} be collection of $H \subset \mathbb{R}^n$ s.t. there exists A open containing H and a C^1 function $f : A \to \mathbb{R}$ s.t.*

$$f(x) = 0 \quad and \quad Df(x) \ne 0 \qquad \forall \, x \in \overline{H}$$

One needs a criterion to determine sets of Γ_{n-1}.

- Recall Whitney Extension Theorem

Lemma 2.2.2. *Let $C \subset \mathbb{R}^n$ closed and $f : C \to \mathbb{R}$, $\nu : C \to \mathbb{R}^n$ be continuous. If for each compact $K \subset C$*

$$\sup \left\{ \frac{|f(y) - f(x) - \nu(x) \cdot (y - x)|}{|x - y|} \mid 0 < |x - y| \leq \delta, \ x, \ y \in K \right\} \to 0 \qquad as \ \delta \to 0$$

Then there exists a global C^1 function $\overline{f} : \mathbb{R}^n \to \mathbb{R}$ s.t. $\overline{f} = f$ and $D\overline{f} = \nu$ on C.

- One hence obtain criterion

  Lemma 2.2.3. *Let $C \subset \mathbb{R}^n$ compact. If there exists $\nu : C \to \mathbb{R}^n$ continuous s.t. $\nu \neq 0$ and*

  $$\lim_{|x-y| \to 0} \frac{\langle \nu(x), \ x - y \rangle}{|x - y|} = 0 \qquad uniformly \ for \ x, \ y \in C \tag{2.30}$$

  Then $C \in \Gamma_{n-1}$.

  Proof. Apply Whitney Extension Theorem with C compact and function $= 0$. Then there exists C^1 function $f : \mathbb{R}^n \to \mathbb{R}$ s.t. $f = 0$ and $Df = \nu$ on C. Since $\nu \neq 0$, conclude $C \in \Gamma_{n-1}$. $\qquad\qquad\square$

Before we prove the main theorem for this section, one needs 2 lemmas to reduce our problem

Lemma 2.2.4 (Egoroff Theorem)**.** *Let μ be measure on $\mathbb{R}^n$ and f_k, $f : \mathbb{R}^n \to \mathbb{R}^m$ μ-measurable. Let $A \subset \mathbb{R}^n$ μ-measurable with $\mu(A) < \infty$ and $f_k \to f$ μ-a.e. on A. Then for any $\varepsilon > 0$, there exists $B \subset A$ μ-measurable s.t. $\mu(A \setminus B) < \varepsilon$ and $f_k \to f$ uniformly on B.*

Lemma 2.2.5 (Lusin Theorem)**.** *Let μ be Borel regular measure on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ μ-measurable. Let $A \subset \mathbb{R}^n$ μ-measurable and $\mu(A) < \infty$. Then for any $\varepsilon > 0$, there exists $K \subset A$ compact s.t. $\mu(A \setminus K) < \varepsilon$ and $f|_K$ is continuous.*

Theorem 2.2.1. *If $E \subset \mathbb{R}^n$ is Caccioppoli, then*

-
  $$\partial^* E = \bigcup_{i=1}^{\infty} C_i \cup N \tag{2.31}$$

  where N is $|D\varphi_E|$-measure zero and $C_i \in \Gamma_{n-1}$ compact for all i.

- *For each Borel set $B \subset \partial^* E$,*
  $$\int_B |D\varphi_E| = H_{n-1}(B) \tag{2.32}$$

  and moreover, for every open set $\Omega \subset \mathbb{R}^n$

  $$P(E, \Omega) = \int_\Omega |D\varphi_E| = H_{n-1}(\partial^* E \cap \Omega) = H_{n-1} \llcorner \partial^* E(\Omega) \tag{2.33}$$

- *Reduced Boundary is dense in ∂E*
  $$\overline{\partial^* E} = \partial E \tag{2.34}$$

Proof. Given E Caccioppoli

- 1. Since E is not necessarily bounded (in particular, $|D\varphi_E|(^*E)$ isn't necessarily finite), one make use of σ-finite of $|D\varphi_E|$ to partition $\mathbb{R}^n = \bigcup_{i=1}^{\infty} \Omega_i$ into open bounded disjoint domains so $|D\varphi_E|(\Omega_i) < \infty$. Then $\partial^* E = \bigcup_{i=1}^{\infty} (\partial^* E \cap \Omega_i)$. If (2.31) is proved for all bounded Caccioppoli sets, then $\partial^* E \cap \Omega_i = \bigcup_j C_{i,j} \cup N_i$ so $\partial^* E = \bigcup_i \bigcup_j C_{i,j} \cup N_i$ is still countable union with $|D\varphi_E|(\bigcup_i N_i) \leq \sum_i |D\varphi_E|(N_i) = 0$. Hence it suffices to prove (2.31) for bounded Caccioppoli Sets.

  2. Given E bounded Caccioppoli, $\partial^* E$ is bounded so $|D\varphi_E|(\partial^* E) < \infty$. Let $|D\varphi_E|$ be our finite measure on the space $\partial^* E$, and denote for any $x \in \partial^* E$, the limits (2.23) and (2.24)

     $$\lim_{\rho \to 0} \frac{1}{\rho^n} |E \cap B_\rho(x) \cap T^-(x)| = 0 \qquad \lim_{\rho \to 0} \frac{1}{\rho^n} |(B_\rho(x) \setminus E) \cap T^+(x)| = 0 \tag{2.35}$$

     as our pointwise limiting function sequences. Then Egoroff theorem applies, saying for any $i > 0$, there exists $F_i \subset \partial^* E$ that is $|D\varphi_E|$-measurable s.t.

     $$|D\varphi_E|(\partial^* E \setminus F_i) < \frac{1}{2i} \tag{2.36}$$

and (2.35) holds uniformly on F_i. Now to apply Lusin theorem, first notice one has $\nu(x) = \lim_{\rho \to 0} \frac{\int_{B_\rho(x)} D\varphi_E}{\int_{B_\rho(x)} |D\varphi_E|}$ defined on $\partial^* E$ as in (2.4). As Radon measure, $|D\varphi_E|$ is Borel-regular, and with $|D\varphi_E|(F_i) < \infty$, Lusin's theorem says there exists $C_i \subset F_i$ compact s.t.

$$|D\varphi_E|(F_i \setminus C_i) < \frac{1}{2i} \tag{2.37}$$

and $\nu|_{C_i}$ is continuous. Thus $\partial^* E = (\partial^* E \setminus (\bigcup_i C_i)) \cup \bigcup_i C_i$ where

$$|D\varphi_E|(\partial^* E \setminus (\bigcup_i C_i)) = \lim_{i \to \infty} \mu(\partial^* E \setminus C_i) = 0$$

so $N = \partial^* E \setminus (\bigcup_i C_i)$ is our $|D\varphi_E|$-measure zero set. It suffices to show $C_i \in \Gamma_{n-1}$.

3. Fix any such C_i. By Egoroff, convergence (2.35) happens on C_i uniformly, so for any $\varepsilon \in (0,1)$, there exists $\sigma \in (0,1)$ s.t. for any $\rho < 2\sigma$

$$|E \cap B_\rho(z) \cap T^-(z)| < \rho^n(\frac{1}{4}\omega_n 2^{-n})\varepsilon^n \qquad uniformly\ on\ z \in C_i \tag{2.38}$$

$$|E \cap B_\rho(z) \cap T^+(z)| = |B_\rho(z) \cap T^+(z)| - |(B_\rho(z) \setminus E) \cap T^+(z)| = \rho^n \omega_n \frac{1}{2} - |(B_\rho(z) \setminus E) \cap T^+(z)|$$

$$> \rho^n \omega_n \frac{1}{2} - \rho^n(\frac{1}{4}\omega_n 2^{-n})\varepsilon^n = \frac{\omega_n \rho^n}{2}(1 - \frac{\varepsilon^n}{2^{n+1}}) \qquad uniformly\ on\ z \in C_i \tag{2.39}$$

Using (2.38) and (2.39), we wish to show for any $\varepsilon \in (0,1)$, there exists $\sigma > 0$ s.t. $\frac{|\langle \nu(x), x-y \rangle|}{|x-y|} < \varepsilon$ uniformly for any $x, y \in C_i$ s.t. $|x - y| < \sigma$. Since C_i are compact, ν is continuous on C_i by Lusin, and by definition for reduced boundary (2.5) that $|\nu(x)| = 1 \neq 0$, applying lemma 2.2.3 concludes that $C_i \in \Gamma_{n-1}$.

4. To show uniform convergence, first suppose there exists some $\varepsilon \in (0,1)$ s.t. for any $\sigma \in (0,1)$, there exists $x, y \in C_i$ s.t. $|x - y| < \sigma$ yet $\langle \nu(x), y - x \rangle < -\varepsilon|x-y|$. By definition of $T^-(x)$, this implies $y \in T^-(x)$ for such x, y. And indeed, for any $|z - y| < \varepsilon|x-y|$, one has

$$\langle \nu(x), z - x \rangle = \langle \nu(x), z - y \rangle + \langle \nu(x), y - x \rangle \leq |z - y| - \varepsilon|x-y| < 0$$

hence $z \in T^-(x)$. Thus $B_{\varepsilon|x-y|}(y) \subset T^-(x)$. Moreover, that $|z - y| < \varepsilon|x-y|$ implies

$$|x - z| \leq |x - y| + |y - z| < |x - y| + \varepsilon|x - y| < 2|x - y|$$

so $B_{\varepsilon|x-y|}(y) \subset B_{2|x-y|}(x)$. Hence we have

$$B_{\varepsilon|x-y|}(y) \subset T^-(x) \cap B_{2|x-y|}(x) \tag{2.40}$$

Now, since we require $|x - y| < \sigma$, choose $\rho = 2|x - y| < 2\sigma$, one may apply (2.38) with $z = x$

$$|E \cap B_{2|x-y|}(x) \cap T^-(x)| < \frac{\omega_n \varepsilon^n}{4}|x - y|^n \tag{2.41}$$

and then use (2.39) with $\rho = \varepsilon|x-y| < \sigma$ and $z = y$

$$|E \cap B_{\varepsilon|x-y|}(y)| \geq |E \cap B_{\varepsilon|x-y|}(y) \cap T^+(y)| > \frac{\omega_n \varepsilon^n |x-y|^n}{2}(1 - \frac{\varepsilon^n}{2^{n+1}}) > \frac{\omega_n \varepsilon^n}{4}|x - y|^n \tag{2.42}$$

But now (2.41) and (2.42) together yields

$$|E \cap B_{2|x-y|}(x) \cap T^-(x)| < |E \cap B_{\varepsilon|x-y|}(y)| \implies B_{2|x-y|}(x) \cap T^-(x) \subsetneq B_{\varepsilon|x-y|}(y)$$

contradicting the inclusion (2.40). Hence our assumption for existence of ε fails if we require that $\langle \nu(x), y - x \rangle < -\varepsilon|x-y|$.

5. Then, we suppose certain $\varepsilon \in (0,1)$ exists and for any $\sigma \in (0,1)$, there exists $x, y \in C_i$ s.t. $|x-y| < \sigma$ yet $\langle \nu(x), y - x \rangle > \varepsilon|x-y|$. One derive the similar inequalities for (2.38) and (2.39) but on $B_\rho(z) \setminus E$

$$|(B_\rho(z) \setminus E) \cap T^+(z)| < \rho^n(\frac{1}{4}\omega_n 2^{-n})\varepsilon^n \qquad uniformly\ on\ z \in C_i \tag{2.43}$$

$$|(B_\rho(z) \setminus E) \cap T^-(z)| = |B_\rho(z) \cap T^-(z)| - |B_\rho(z) \cap E \cap T^-(z)| = \rho^n \omega_n \frac{1}{2} - |B_\rho(z) \cap E \cap T^-(z)|$$

$$> \rho^n \omega_n \frac{1}{2} - \rho^n(\frac{1}{4}\omega_n 2^{-n})\varepsilon^n = \frac{\omega_n \rho^n}{2}(1 - \frac{\varepsilon^n}{2^{n+1}}) \qquad uniformly\ on\ z \in C_i \tag{2.44}$$

By definition of $T^+(x)$, $y \in T^+(x)$, and indeed for any $|z - y| < \varepsilon|x - y|$

$$\langle \nu(x), z - x \rangle = \langle \nu(x), z - y \rangle + \langle \nu(x), y - x \rangle > -\varepsilon|x - y| + \varepsilon|x - y| > 0$$

hence $z \in T^+(x)$. Thus $B_{\varepsilon|x-y|}(y) \subset T^+(x)$. Moreover, that $|z - y| < \varepsilon|x - y|$ implies $B_{\varepsilon|x-y|}(y) \subset B_{2|x-y|}(x)$. Hence we have

$$B_{\varepsilon|x-y|}(y) \subset T^+(x) \cap B_{2|x-y|}(x) \tag{2.45}$$

Apply (2.43) with $z = x$ and $\rho = 2|x - y| < 2\sigma$ yields

$$|(B_{2|x-y|}(x) \setminus E) \cap T^+(x)| < \frac{\omega_n \varepsilon^n}{4}|x - y|^n \tag{2.46}$$

Then use (2.44) with $z = y$ and $\rho = \varepsilon|x - y| < \sigma$

$$|B_{\varepsilon|x-y|}(y) \setminus E| \geq |(B_{\varepsilon|x-y|}(y) \setminus E) \cap T^-(y)| > \frac{\omega_n \varepsilon^n}{4}|x - y|^n \tag{2.47}$$

together (2.46) and (2.47) yields

$$T^+(x) \cap B_{2|x-y|}(x) \subsetneq B_{\varepsilon|x-y|}(y)$$

and we have contradiction.

6. We conclude that $\frac{\langle \nu(x), x-y \rangle}{|x-y|}$ converges to 0 uniformly for $x, y \in C_i$ for any i. Hence by lemma 2.2.3, $C_i \in \Gamma_{n-1}$ for any i. Thus we've proved (2.31).

- 1. For any $B \subset \partial^* E$, $B = (B \cap (\bigcup_i C_i)) \cup (B \setminus \bigcup_i C_i)$, but by continuity from above and the Ratio Estimate (2.27)

$$H_{n-1}(B \setminus \bigcup_i C_i) = \lim_{i \to \infty} H_{n-1}(B \cap C_i^c) \leq \lim_{i \to \infty} 2 \cdot 3^{n-1} \int_{B \setminus C_i} |D\varphi_E| \leq 2 \cdot 3^{n-1} \lim_{i \to \infty} \frac{1}{i} = 0$$

where (2.36) and (2.37) gives

$$\int_{B \setminus C_i} |D\varphi_E| \leq \int_{\partial^* E \setminus C_i} |D\varphi_E| \leq \int_{\partial^* E \setminus F_i} |D\varphi_E| + \int_{F_i \setminus C_i} |D\varphi_E| < \frac{1}{i}$$

Hence $H_{n-1}(B) = H_{n-1}(\bigcup_i (B \cap C_i))$ and it suffices to prove (2.32) for $B \cap C_i$. Moreover, subsets of $C_i \in \Gamma_{n-1}$ still belong to Γ_{n-1}, so it suffices to prove for $B \in \Gamma_{n-1}$.

- 2. Given $B \in \Gamma_{n-1}$, there exists open set $A \supset \overline{B}$ and $f : A \to \mathbb{R}$ that is C^1 s.t.

$$f(x) = 0 \qquad and \qquad Df(x) \neq 0 \qquad \forall x \in \overline{B}$$

Since f is C^1, up to taking subset, one may assume $Df(x) \neq 0$ on the open set A. Now $0 \in \mathbb{R}$ is a regular value for the map $f : A \to \mathbb{R}$, so taking its preimage

$$V := \{x \in A \mid f(x) = 0\}$$

one has from preimage theorem that V is regular hypersurface of dimension $n - 1$.

- 3. From Hausdorff measure properties, $H_{n-1} \llcorner V$ has density for any $x \in B$

$$\lim_{\rho \to 0} \frac{H_{n-1} \llcorner V(B_\rho(x))}{\rho^{n-1}} = \omega_{n-1} \qquad for\ any\ x \in B$$

and because $x \in B \subset \partial^* E$, taking $\varepsilon = 1$ in density estimates for $|D\varphi_E|$ in (2.25) gives

$$\lim_{\rho \to 0} \frac{\int_{B_\rho(x)} |D\varphi_E|}{\rho^{n-1}} = \omega_{n-1} \qquad for\ any\ x \in B$$

hence $\lim_{\rho \to 0} \frac{H_{n-1} \llcorner V(B_\rho(x))}{\int_{B_\rho(x)} |D\varphi_E|} = 1$ for any $x \in B$. Now by Lebesgue-Besicovitch differentiation 2.1.2 and Ratio estimate (2.27) implying $H_{n-1} \ll |D\varphi_E|$ on $\partial^* E$, and that $B \subset V$

$$H_{n-1}(B) = H_{n-1} \llcorner V(B) = \lim_{\rho \to 0} \frac{H_{n-1} \llcorner V(B_\rho(x))}{\int_{B_\rho(x)} |D\varphi_E|} \cdot \int_B |D\varphi_E| = \int_B |D\varphi_E| \qquad \forall\ B \subset \partial^* E\ Borel$$

concluding (2.32).

4. From (2.32), take $B = \partial^* E \cap \Omega$ for any $\Omega \subset \mathbb{R}^n$ open gives

$$H_{n-1} \llcorner \partial^* E(\Omega) = H_{n-1}(\partial^* E \cap \Omega) = \int_{\partial^* E \cap \Omega} |D\varphi_E|$$

But according to Lebesgue-Besicovitch differentiation 2.1.2, in particular, remark 2.1.1, one has $|D\varphi_E|(\partial E \setminus \partial^* E) = 0$. Hence the above writes, upon using $\operatorname{supp}(D\varphi_E) \subset \partial E$

$$H_{n-1} \llcorner \partial^* E(\Omega) = \int_{\partial E \cap \Omega} |D\varphi_E| = \int_{\Omega} |D\varphi_E| = P(E, \Omega)$$

concluding (2.33)

- To show density, one show that for any A open set intersecting ∂E, it must also intersect $\partial^* E$. In other words, if A open not does intersect $\partial^* E$, by (2.33)

$$0 = H_{n-1}(\partial^* E \cap A) = \int_A |D\varphi_E|$$

but since A is open, ∂E is closed, they do not intersect. Hence $\overline{\partial^* E} = \partial E$.

$\square$