[Giusti] Minimial Surfaces and Functions of Bounded Variation

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Contents

Chapter 1

Functions of Bounded Variation

1.1 Functions of Bounded Variation and Caccioppoli Sets

1.1.1 Definitions and Semicontinuity

Definition 1.1.1 (BV Functions). *Let $\Omega \subset \mathbb{R}^n$ be open set. $f \in L^1(\Omega)$.*

$$\int_\Omega |Df| := \sup\{\int_\Omega f \operatorname{div} g \, dx \mid g \in C_0^1(\Omega; \mathbb{R}^n), |g(x)| \leq 1\} \tag{1.1}$$

$f \in BV(\Omega)$ if $\int_\Omega |Df| < \infty$. $BV(\Omega)$ is space of $L^1(\Omega)$ functions of bounded variation in Ω.

Example 1.1.1. *If $f \in C^1(\Omega)$, $\int_\Omega |Df| = \int_\Omega |\nabla f| dx$ where $\nabla f \in C(\Omega; \mathbb{R}^n)$ is classical gradient. If $f \in W^{1,1}(\Omega)$, $\int_\Omega |Df| = \int_\Omega |\nabla f| dx$ where $\nabla f \in L^1(\Omega; \mathbb{R}^n)$ is weak gradient.*

Example 1.1.2. *We study $\varphi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \in \mathbb{R}^n \setminus E \end{cases}$ characteristic on E with C^2 boundary.*

- *If E is bounded, $\|\varphi_E\|_{L^1(\Omega)} = |E \cap \Omega| < \infty$ so $\varphi_E \in L^1(\Omega)$. But $\nabla \varphi_E$ distributional derivative is vector-valued Radon measure instead of $L^1(\Omega)$ function, hence $\varphi_E \notin W^{1,1}(\Omega)$. But on the other hand, we may compute $\int_\Omega |D\varphi_E|$. Let $g \in C_0^1(\Omega; \mathbb{R}^n)$ s.t. $|g| \leq 1$, so by Gauss-Green formula*

$$\int_\Omega \varphi_E \operatorname{div} g \, dx = \int_E \operatorname{div} g \, dx = \int_{\partial E} g \cdot \nu \, dH_{n-1} \leq H_{n-1}(\partial E \cap \Omega) \tag{1.2}$$

  for ν outer unit normal to ∂E. Taking supremum in g yields $\int_\Omega |D\varphi_E| < \infty$. Thus $W^{1,1}(\Omega) \subsetneq BV(\Omega)$.

- *We in fact prove $\int_\Omega |D\varphi_E| = H_{n-1}(\partial E \cap \Omega)$. Since E C^2 boundary, $v \in C^1(\partial E; \mathbb{R}^n)$ with $|\nu| = 1$. Since ∂E is closed in $\mathbb{R}^n$ and $\mathbb{R}^n$ is normal, we may apply Tietze Extension to extend ν to $N \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ with $|N| \leq 1$. By Urysohn's there exists $\eta \in C_0^\infty(\Omega)$ s.t. $|\eta| \leq 1$, so let $g = \eta N \in C_0^1(\Omega; \mathbb{R}^n)$*

$$\int_\Omega \varphi_E \operatorname{div} g \, dx = \int_E \operatorname{div} g \, dx = \int_{\partial E} \eta N \cdot \nu \, dH_{n-1} = \int_{\partial E} \eta \, dH_{n-1}$$

  Take supremum in g on LHS and in η on RHS yields (due to $H_{n-1} \llcorner \partial E$ is Radon measure on $\mathbb{R}^n$)

$$\int_\Omega |D\varphi_E| \geq \sup\{\int_{\partial E} \eta \, dH_{n-1} \mid \eta \in C_0^\infty(\Omega), |\eta| \leq 1\} = H_{n-1}(\partial E \cap \Omega) \tag{1.3}$$

  Hence (1.2) and (1.3) together gives, for E C^2 boundary

$$\int_\Omega |D\varphi_E| = H_{n-1}(\partial E \cap \Omega) \tag{1.4}$$

Remark 1.1.1. *For $f \in BV(\Omega)$, the duality pairing $\langle Df, g \rangle := -\int_\Omega f \operatorname{div} g \, dx$ defines the distributional gradient $Df \in (C_0^1(\Omega; \mathbb{R}^n))'$ because $\int_\Omega |Df| = \sup_{g \in C_0^1(\Omega; \mathbb{R}^n)} \frac{|\langle Df, g \rangle|}{|g|} < \infty$. By Riesz, the bounded linear functional Df on $C_0^1(\Omega; \mathbb{R}^n)$ defines a vector-valued Radon measure Df on Ω with $\int_\Omega |Df|$ the total variation of Df on Ω. Since $|Df|$ is a Borel measure over Ω, one may measure $\int_A |Df|$ for $A \subset \Omega$ not necessarily open. In particular, if $f = \varphi_E$ for some E bounded and C^2 so that $\varphi_E \in BV(\Omega)$, since the two Borel measures $|D\varphi_E|$ and $H_{n-1} \llcorner \partial E$ agrees on all open sets as in (1.4), they agree on all Borel sets.*

Definition 1.1.2 (Perimeter & Caccioppoli Set). *Let $\Omega \subset \mathbb{R}^n$ be open and E a Borel set. The Perimeter of E in Ω is*

$$P(E,\Omega) := \int_\Omega |D\varphi_E| = \sup\{\int_E \text{div} g \, dx \mid g \in C_0^1(\Omega;\mathbb{R}^n), |g| \leq 1\} \tag{1.5}$$

If $\Omega = \mathbb{R}^n$ write $P(E) := P(E,\mathbb{R}^n)$. The Borel set E is a Caccioppoli Set if it has locally finite perimeter, i.e., $P(E,\Omega) < \infty$ for each bounded open $\Omega \subset \mathbb{R}^n$.

Remark 1.1.2. *One has characterisations for Caccioppoli Sets E*

- *E is a Caccioppoli Set iff there exist vector-valued Radon measure ω over $\mathbb{R}^n$ s.t.*

  1. *ω has locally finite variation, i.e., for each bounded open $\Omega \subset \mathbb{R}^n$, $|\omega|(\Omega) < \infty$*
  2. *for all $g \in C_0^1(\Omega;\mathbb{R}^n)$ s.t. $|g| \leq 1$, one has $\int_E \text{div} g \, dx = \int g \cdot d\omega$*

  Proof. $\Longrightarrow$ Since for each Ω bounded and open, $P(E,\Omega) = \int_\Omega |D\varphi_E| < \infty$ iff $\varphi_E \in BV(\Omega)$, $D\varphi_E$ defines a vector-valued Radon measure with locally finite variation over $\mathbb{R}^n$. Let $\omega = -D\varphi_E$, so for each fixed Ω,

  $$\int g \cdot d\omega = -\langle D\varphi_E, g\rangle = \int_\Omega \varphi_E \, \text{div} g \, dx = \int_E \text{div} g \, dx$$

  $\Longleftarrow$ Suppose such ω exists. Then for any $g \in C_0^1(\Omega;\mathbb{R}^n)$ s.t. $|g| \leq 1$

  $$\int_E \text{div} g \, dx = \int g \cdot d\omega \leq |\omega|(\Omega) < \infty$$

  take supremum in g on LHS gives $P(E,\Omega) = \int_\Omega |D\varphi_E| \leq |\omega|(\Omega) < \infty$. □

- *For E any Borel Set,* $\text{supp} D\varphi_E \subset \partial E$ *where*

  $$\text{supp} D\varphi_E := \mathbb{R}^n \setminus \bigcup \left\{ A \textit{ open} \mid \forall\, g \in C_0^1(A;\mathbb{R}^n),\, |g| \leq 1 \implies \int g \cdot D\varphi_E = 0 \right\}$$

  Proof. For any $x \notin \partial E$, there exists A open neighbor of x s.t. either $A \subset E$ or $A \subset E^c$. If $A \subset E^c$, $\varphi_E = 0$ on A, so for any $g \in C_0^1(A;\mathbb{R}^n)$, $|g| \leq 1$ one indeed has $\int g \cdot D\varphi_E = -\int \varphi_E \, \text{div} g \, dx = 0$. If $A \subset E$, $\varphi_E = 1$ on A, so for such g, $\int g \cdot D\varphi_E = -\int_E \text{div} g \, dx = -\int \text{div} g \, dx = 0$ since g is compactly supported and one apply the divergence theorem. Thus for any $x \notin \partial E$, $x \notin \text{supp} D\varphi_E$. □

- *E is a Caccioppoli Set iff the Gauss-Green formula holds in a generalized sense, i.e., for any $\Omega \subset \mathbb{R}^n$ open and bounded, and for any $g \in C_0^1(\Omega;\mathbb{R}^n)$ s.t. $|g| \leq 1$*

  $$\int_E \text{div} g \, dx = -\int_{\partial E} g \cdot D\varphi_E \tag{1.6}$$

  Proof. $\Longrightarrow$ follows directly. $\Longleftarrow$ By the previous item, $\int_{\partial E} g \cdot D\varphi_E = \int g \cdot D\varphi_E$. Indeed, $\omega := -D\varphi_E$ has bounded variation on each open bounded Ω. Use the first item that characterises Caccioppoli set. □

- *Given Caccioppoli set E, one has useful identification of $\varphi_E \in BV$*

  Corollary 1.1.1. *For E Caccioppoli, and $\Omega \subset \mathbb{R}^n$ open. If either E or Ω is bounded, $\varphi_E \in BV(\Omega)$.*

  Proof. Since either E or Ω is bounded, $\|\varphi_E\|_{L^1(\Omega)} = |E \cap \Omega| < \infty$ hence $\varphi_E \in L^1(\Omega)$. Now one compute $\int_\Omega |D\varphi_E|$, and may proceed in 2 directions. If Ω itself is bounded, since E Caccioppoli gives locally finite perimeter, indeed $\int_\Omega |D\varphi_E| < \infty$. If on the other hand, E is bounded, for any $g \in C_0^1(\Omega;\mathbb{R}^n)$ s.t. $|g| \leq 1$, using (1.6)

  $$\int_\Omega \varphi_E \, \text{div} g \, dx = \int_E \text{div} g \, dx = -\int_{\partial E} g \cdot D\varphi_E$$

  ∂E is bounded and closed, hence compact. Then one may cover ∂E using sufficient large open ball B_R, and since E is Caccioppoli, $|D\varphi_E|$ defines locally finite variation positive measure

  $$-\int_{\partial E} g \cdot D\varphi_E \leq \int_{B_R \cap \Omega} |D\varphi_E| < \infty$$

  □

Theorem 1.1.1 (Semi-continuity). *Let $\Omega \subset \mathbb{R}^n$ open. $\{f_j\} \subset BV(\Omega)$ s.t. $f_j \to f$ in $L^1_{loc}(\Omega)$, then*

$$\int_\Omega |Df| \le \liminf_{j\to\infty} \int_\Omega |Df_j| \tag{1.7}$$

Proof. For any $g \in C^1_0(\Omega;\mathbb{R}^n)$ s.t. $|g| \le 1$

$$\int_\Omega f \operatorname{div} g \, dx = \lim_{j\to\infty} \int_\Omega f_j \operatorname{div} g \, dx \le \liminf_{j\to\infty} |Df_j|$$

take supremum in g on LHS. □

Remark 1.1.3. *The equality in (1.7) may not be achieved. Let $\Omega = (0, 2\pi)$ and $f_j(x) = \frac{1}{j}\sin(jx)$. Note $\int_0^{2\pi} |\frac{1}{j}\sin(jx)|dx \le 2\pi\frac{1}{j} \to 0$ so $f_j \to 0$ in $L^1(0, 2\pi)$. But $f_j'(x) = \cos(jx)$ and $\int_0^{2\pi} |Df_j| = \int_0^{2\pi} |\cos(jx)|dx = 4$.*

Proposition 1.1.1. *For $\Omega \subset \mathbb{R}^n$ open, $BV(\Omega)$ with norm $\|f\|_{BV} := \|f\|_{L^1} + \int_\Omega |Df|$ is a Banach Space.*

Proof. That $\|f\|_{BV}$ defines a norm follows from L^1 norm and homogeneity, subadditivity of total variation. To see $BV(\Omega)$ is complete, take Cauchy sequence $\{f_j\}$ in $BV(\Omega)$. Since $\{f_j\}$ is already Cauchy in $L^1(\Omega)$, there exists $f \in L^1(\Omega)$ s.t. $\|f - f_j\|_{L^1} \to 0$. Also, there exits N s.t. $\forall\, m, n \ge N$, $\int_\Omega |D(f_m - f_n)| \le 1$, one has $\int_\Omega |Df_j| \le \max_{1\le i\le N} \int_\Omega |Df_i| + 1$ uniformly bounded. Hence (1.7) semicontinuity gives $\int_\Omega |Df| < \infty$ so $f \in BV(\Omega)$. It suffices to show $\int_\Omega |D(f - f_j)| \to 0$. For any $\varepsilon > 0$, there exists N s.t. for any $j, k \ge N$, $\int_\Omega |D(f_j - f_k)| \le \varepsilon$. Fix j, apply (1.7) semicontinuity to $\{f_j - f_k\}_k$ so $\int_\Omega |D(f_j - f)| \le \liminf_{k\to\infty} \int_\Omega |D(f_j - f_k)| \le \varepsilon$. Take ε to 0. □

Proposition 1.1.2. *Let $\Omega \subset \mathbb{R}^n$ open. $f, f_j \in BV(\Omega)$ s.t. $f_j \to f$ in $L^1_{loc}(\Omega)$ and $\int_\Omega |Df| = \lim_{j\to\infty} \int_\Omega |Df_j|$. Then for any $A \subset \Omega$ open, one has certain reverse direction to (1.7)*

$$\int_{\overline{A}\cap\Omega} |Df| \ge \limsup_{j\to 0} \int_{\overline{A}\cap\Omega} |Df_j|$$

in particular, if $\int_{\partial A\cap\Omega} |Df| = 0$, one has

$$\int_A |Df| = \lim_{j\to 0} \int_A |Df_j| \tag{1.8}$$

Proof. Let $B := \Omega \setminus \overline{A}$ so $B \subset \Omega$ open. By semicontinuity (1.7)

$$\int_A |Df| \le \liminf_{j\to 0} \int_A |Df_j| \qquad \int_B |Df| \le \liminf_{j\to 0} \int_B |Df_j|$$

one calculate

$$\begin{aligned}\int_{\overline{A}\cap\Omega} |Df| + \int_B |Df| &= \int_\Omega |Df| = \lim_{j\to\infty} \int_\Omega |Df_j| \\ &\ge \limsup_{j\to 0} \int_{\overline{A}\cap\Omega} |Df_j| + \liminf_{j\to\infty} \int_B |Df_j| \ge \limsup_{j\to 0} \int_{\overline{A}\cap\Omega} |Df_j| + \int_B |Df|\end{aligned}$$

since $f \in BV(\Omega)$, indeed $\int_B |Df| < \infty$ so one may cancel out. To see (1.8), one notice $A \subset \Omega$. □

1.1.2 Approximation by smooth functions

Definition 1.1.3. *$\eta(x)$ is mollifier if $\begin{cases} \eta \in C_0^\infty(\mathbb{R}^n) \\ \operatorname{supp} \eta \subset B_1 \\ \int \eta \, dx = 1 \end{cases}$ If moreover, $\begin{cases} \eta \ge 0 \\ \eta(x) = \mu(|x|) \end{cases}$ η is positive symmetric.*

Standard example for such positive symmetric mollifier is $\eta = \frac{1}{\int \gamma dx}\gamma$ where $\gamma(x) := \begin{cases} 0 & |x| \ge 1 \\ \exp(\frac{1}{|x|^2-1}) & |x| < 1 \end{cases}$

Definition 1.1.4. *Given a positive symmetric mollifier η, the rescaled mollifier $\eta_\varepsilon(x) := \frac{1}{\varepsilon^n}\eta(\frac{x}{\varepsilon})$ satisfies $\operatorname{supp} \eta_\varepsilon \subset B_\varepsilon$. Given $f \in L^1_{loc}(\Omega)$, define its mollification $f_\varepsilon := \eta_\varepsilon * f$*

$$f_\varepsilon(x) = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} \eta(\frac{x-y}{\varepsilon}) f(y) \, dy = (-1)^n \int_{\mathbb{R}^n} \eta(z) f(x - \varepsilon z) \, dz = \int_{\mathbb{R}^n} \eta(z) f(x + \varepsilon z) \, dz \tag{1.9}$$

Lemma 1.1.1. *One has tools from mollification*

- $f_\varepsilon \in C^\infty(\mathbb{R}^n)$, $f_\varepsilon \to f$ *in* $L^1_{loc}(\Omega)$. *If* $f \in L^1(\Omega)$, $f_\varepsilon \to f$ *in* $L^1(\Omega)$.

- *If $A \le f(x) \le B$ for any $x \in \Omega$, then $A \le f_\varepsilon(x) \le B$ for any $x \in \Omega$.*
- *If $f, g \in L^1(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} f_\varepsilon g dx = \int_{\mathbb{R}^n} f g_\varepsilon dx$.*
- *If $f \in C^1(\mathbb{R}^n)$, then $(\frac{\partial}{\partial x_i} f)_\varepsilon = \frac{\partial}{\partial x_i}(f_\varepsilon)$ for $i = 1, \cdots, n$.*
- $\operatorname{supp} f := \overline{\{x \in \mathbb{R}^n \mid f \neq 0\}} \subset A$, *then* $\operatorname{supp} f_\varepsilon \subset A_\varepsilon := \{x \mid \operatorname{dist}(x, A) \le \varepsilon\}$.

Proposition 1.1.3. *$\Omega \subset \mathbb{R}^n$ open, $f \in BV(\Omega)$. For $A \subset\subset \Omega$ open s.t. $\int_{\partial A} |Df| = 0$, one has*

$$\int_A |Df| = \lim_{\varepsilon \to 0} \int_A |Df_\varepsilon| dx \tag{1.10}$$

Proof. Since $f \in L^1(\Omega)$, $f_\varepsilon \to f$ in $L^1(\Omega)$, by semicontinuity (1.7), one has $\int_A |Df| \le \liminf_{\varepsilon \to 0} \int_A |Df_\varepsilon|$. It suffices to prove $\int_A |Df| \ge \limsup_{\varepsilon \to 0} \int_A |Df_\varepsilon|$. For any $g \in C_0^1(A; \mathbb{R}^n)$ s.t. $|g| \le 1$, using tools from mollification

$$\int_A f_\varepsilon \operatorname{div} g \, dx = \int_A f (\operatorname{div} g)_\varepsilon \, dx = \int_A f \operatorname{div}(g_\varepsilon) \, dx$$

$|g| \le 1 \implies |g_\varepsilon| \le 1$ and $\operatorname{supp} g \subset A \implies \operatorname{supp} g_\varepsilon \subset A_\varepsilon$. Hence taking supremum in g

$$\int_A |Df_\varepsilon| \le \int_{A_\varepsilon} |Df|$$

Take $\limsup$ on LHS and use continuity from above on RHS ($f \in BV(\Omega)$ defines a Radon measure $|Df|$)

$$\limsup_{\varepsilon \to 0} \int_A |Df_\varepsilon| \le \lim \int_{A_\varepsilon} |Df| = \int_{\overline{A}} |Df|$$

Now by our assumption, RHS equals $\int_A |Df|$. ∎

Remark 1.1.4. *Note in (1.10) we require $A \subset\subset \Omega$ not because we need boundedness, but because we wish that A and A_ε do not touch $\partial\Omega$. And this problem is resolved for taking $\Omega = \mathbb{R}^n$, and indeed, one may do so for $A = A_\varepsilon = \mathbb{R}^n$ ($\partial A = \partial \mathbb{R}^n = \varnothing$). Now for any $f \in BV(\mathbb{R}^n)$, one has*

$$\int_{\mathbb{R}^n} |Df| = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} |Df_\varepsilon| \, dx \tag{1.11}$$

Indeed for E bounded Caccioppoli, $\varphi_E \in BV(\mathbb{R}^n)$ by Corollary 1.1.1, so (1.11) applies to φ_E.

(1.10) motivates our approximation of $f \in BV(\Omega)$ using smooth functions. Note approximation in BV norm should not be expected since the BV-closure of $C^\infty(\Omega)$ is $W^{1,1}(\Omega) \subsetneq BV(\Omega)$.

Theorem 1.1.2 (Approximation using C^∞)**.** *$\Omega \subset \mathbb{R}^n$ open, $f \in BV(\Omega)$. There exists $\{f_j\} \subset C^\infty(\Omega)$ s.t.*

$$\lim_{j \to \infty} \int_\Omega |f_j - f| \, dx = 0 \tag{1.12}$$

$$\lim_{j \to 0} \int_\Omega |Df_j| dx = \int_\Omega |Df| \tag{1.13}$$

Proof. Since $f \in BV(\Omega)$, $|Df|$ on Ω is finite measure, so $\forall\, \varepsilon > 0$, there exists $m \in \mathbb{N}$ s.t. $\int_{\Omega \setminus \Omega_0} |Df| < \varepsilon$ where

$$\Omega_k := \left\{ x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \frac{1}{m+k} \right\} \qquad k \ge 0 \tag{1.14}$$

Define sequence $\{A_i\}_{i \ge 1}$ s.t. $A_1 := \Omega_2$, $A_i := \Omega_{i+1} \setminus \overline{\Omega}_{i-1}$ for $i \ge 2$. Note A_i are open and $\Omega \subset \bigcup_{i \ge 1} A_i$. There exists smooth partition of unity $\{\phi_i\}$ subordinate to the cover $\{A_i\}$ s.t.

$$\phi_i \in C_0^\infty(A_i), \quad 0 \le \phi_i \le 1, \quad \sum_{i=1}^\infty \phi_i = 1$$

Note for any $x \in \Omega$, at most 2 of the A_i covers x, hence $\sum_i \phi_i$ is finite sum pointwise, thus $f = \sum_{i=1}^\infty f\phi_i$. One wish to construct certain mollification of f so that our desired approximation holds, and a common method is to mollify each $f\phi_i$ with ε_i chose for each $i \ge 1$ then sum them up. Each ε_i needs to satisfy (let $\Omega_{-1} := \varnothing$)

$$\operatorname{supp}(\eta_{\varepsilon_i} * (f\phi_i)) \subset \Omega_{i+2} \setminus \overline{\Omega}_{i-2} \tag{1.15}$$

$$\|\eta_{\varepsilon_i} * (f\phi_i) - f\phi_i\|_{L^1(\Omega)} < \varepsilon/2^i \tag{1.16}$$

$$\|\eta_{\varepsilon_i} * (fD\phi_i) - fD\phi_i\|_{L^1(\Omega)} < \varepsilon/2^i \tag{1.17}$$

and define $f_\varepsilon := \sum_{i=1}^{\infty} \eta_{\varepsilon_i} * (f\phi_i)$. Note $f_\varepsilon \in C^\infty(\Omega)$ since at each $x \in \Omega$, at most 4 supports from (1.15) covers x, hence finite sum of smooth functions gives smoothness. One immediately has from (1.16)

$$\int_\Omega |f_\varepsilon - f| dx \leq \sum_{i=1}^{\infty} \int_\Omega |\eta_i * (f\phi_i) - f\phi_i| \, dx < \varepsilon$$

hence (1.12) holds. And by semicontinuity (1.7), one has $\int_\Omega |Df| \leq \liminf_{\varepsilon \to 0} \int_\Omega |Df_\varepsilon|$. It suffices to prove $\int_\Omega |Df| \geq \limsup_{\varepsilon \to 0} \int_\Omega |Df_\varepsilon|$. For any $g \in C_0^1(\Omega; \mathbb{R}^n)$ s.t. $|g| \leq 1$,

$$\int_\Omega f_\varepsilon \mathrm{div}\, g \, dx = \sum_{i=1}^{\infty} \int_\Omega \eta_{\varepsilon_i} * (f\phi_i) \mathrm{div}\, g \, dx = \sum_{i=1}^{\infty} \int_\Omega f\phi_i \mathrm{div}\, (\eta_{\varepsilon_i} * g) \, dx$$

notice

$$\mathrm{div}(\phi_i \, \eta_{\varepsilon_i} * g) = D\phi_i \cdot (\eta_{\varepsilon_i} * g) + \phi_i \, \mathrm{div}\, (\eta_{\varepsilon_i} * g)$$

hence

$$\begin{aligned}
\int_\Omega f_\varepsilon \mathrm{div}\, g \, dx &= \sum_{i=1}^{\infty} \int_\Omega f \left[\mathrm{div}(\phi_i \, \eta_{\varepsilon_i} * g) - D\phi_i \cdot (\eta_{\varepsilon_i} * g)\right] \, dx \\
&= \int_\Omega f \mathrm{div}(\phi_1 \, \eta_{\varepsilon_1} * g) \, dx + \sum_{i=2}^{\infty} \int_\Omega f \mathrm{div}(\phi_i \, \eta_{\varepsilon_i} * g) \, dx - \sum_{i=1}^{\infty} \int_\Omega f D\phi_i \cdot (\eta_{\varepsilon_i} * g) \, dx \\
&= \int_\Omega f \mathrm{div}(\phi_1 \, \eta_{\varepsilon_1} * g) \, dx + \sum_{i=2}^{\infty} \int_\Omega f \mathrm{div}(\phi_i \, \eta_{\varepsilon_i} * g) \, dx - \sum_{i=1}^{\infty} \int_\Omega \eta_{\varepsilon_i} * (f D\phi_i) \cdot g \, dx
\end{aligned}$$

notice the pointwise finite sum implies

$$\sum_{i=1}^{\infty} \phi_i = 1 \implies \sum_{i=1}^{\infty} D\phi_i = 0$$

hence one may add back the sum of gradients

$$\int_\Omega f_\varepsilon \mathrm{div}\, g \, dx = \int_\Omega f \mathrm{div}(\phi_1 \, \eta_{\varepsilon_1} * g) \, dx + \sum_{i=2}^{\infty} \int_\Omega f \mathrm{div}(\phi_i \, \eta_{\varepsilon_i} * g) \, dx - \sum_{i=1}^{\infty} \int_\Omega [\eta_{\varepsilon_i} * (f D\phi_i) - f D\phi_i] \cdot g \, dx$$

now by direct estimate, (1.15) and (1.17) respectively

$$\begin{aligned}
&\int_\Omega f \mathrm{div}(\phi_1 \, \eta_{\varepsilon_1} * g) \, dx \leq \int_\Omega |Df| \\
&\sum_{i=2}^{\infty} \int_\Omega f \mathrm{div}(\phi_i \, \eta_{\varepsilon_i} * g) \, dx \leq 3 \int_{\Omega \setminus \Omega_0} |Df| < 3\varepsilon \\
&\sum_{i=1}^{\infty} \int_\Omega [\eta_{\varepsilon_i} * (f D\phi_i) - f D\phi_i] \cdot g \, dx < \varepsilon
\end{aligned}$$

Hence taking supremum in g on LHS gives

$$\int_\Omega |Df_\varepsilon| \leq \int_\Omega |Df| + 4\varepsilon \quad \implies \limsup_{\varepsilon \to 0} \int_\Omega |Df_\varepsilon| \leq \int_\Omega |Df|$$

and (1.13) immediately follows. □

Remark 1.1.5 (Boundary Behavior of Smooth Approximation). *$\Omega \subset \mathbb{R}^n$ open, $f \in BV(\Omega)$. For every $\varepsilon > 0$, $N > 0$ and $x_0 \in \partial\Omega$, let f_ε be as above*

$$\lim_{\rho \to 0} \frac{1}{\rho^N} \int_{B_\rho(x_0) \cap \Omega} |f_\varepsilon - f| \, dx = 0 \tag{1.18}$$

Proof. For $\varepsilon > 0$, choose $m \in \mathbb{N}$, Ω_k as in (1.14) and f_ε as in Theorem 1.1.2. One wish to determine i_0 w.r.t. ρ so that for any $x \in B_\rho(x_0) \cap \Omega$, one may write

$$f_\varepsilon(x) - f(x) = \sum_{i=1}^{\infty} (\eta_{\varepsilon_i} * (f\phi_i) - f\phi_i) = \sum_{i=i_0}^{\infty} (\eta_{\varepsilon_i} * (f\phi_i) - f\phi_i)$$

Making use of (1.15), one needs i_0 to be the smallest integer i s.t. $\partial B_\rho(x_0) \cap \Omega$ touches $\mathrm{supp}\eta_{\varepsilon_i} * (f\phi_i)$, i.e.

$$\frac{1}{m + i_0 + 2} \leq \rho \leq \frac{1}{m + i_0 + 1} \implies i_0 = \lceil \frac{1}{\rho} \rceil - m - 2$$

thus via (1.16), for some constant C independent of ρ

$$\int_{B_\rho(x)\cap\Omega} |f_\varepsilon - f|\, dx \le \sum_{i=i_0}^{\infty} \|\eta_{\varepsilon_i} * (f\phi_i) - f\phi_i\|_{L^1(\Omega)} \le C\, 2^{-i_0} = C\, 2^{-\frac{1}{\rho}}$$

where $2^{-\frac{1}{\rho}}$ goes to 0 exponentially fast. Hence multiplying both sides by $\frac{1}{\rho^N}$ and sending $\rho \to 0$ gives (1.18). □

1.1.3 Compactness Theorem and Existence of Minimizing Caccioppoli sets

One shall recall the GNS type Sobolev Embedding and Rellich Theorem from Sobolev Spaces.

Lemma 1.1.2 (Sobolev Embedding). *$\Omega \subset \mathbb{R}^n$ bounded open. $\partial\Omega$ Lipschitz continuous. $1 \le p \le n$. Then*

$$W^{1,p}(\Omega) \subset L^q(\Omega) \qquad \forall\ 1 \le q \le \frac{np}{n-p} \tag{1.19}$$

i.e., for any such $1 \le q \le \frac{np}{n-p}$, there exists $C = C(n, p, q, \Omega)$ s.t.

$$\|f\|_{L^q} \le C\,\|f\|_{W^{1,p}} \tag{1.20}$$

Lemma 1.1.3 (Rellich-Kondrachov). *$\Omega \subset \mathbb{R}^n$ bounded open. $\partial\Omega$ Lipschitz continuous. $1 \le p < n$. Then*

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega) \qquad \forall\ 1 \le q < \frac{np}{n-p} \tag{1.21}$$

i.e., each uniformly bounded sequence $\{f_j\}$ in $W^{1,p}(\Omega)$ norm has a convergent subsequence $\{f_{j_k}\}$ in $L^q(\Omega)$ norm for each $q \in [1, \frac{np}{n-p})$.

Using above lemmas, one may show for the corresponding BV Embedding and a Compactness Theorem.

Theorem 1.1.3 (GNS-type BV Embedding). *$\Omega \subset \mathbb{R}^n$ bounded open. $\partial\Omega$ Lipschitz continuous. Then*

$$BV(\Omega) \subset L^p(\Omega) \qquad \forall\ 1 \le p \le \frac{n}{n-1} \tag{1.22}$$

i.e., for any such $1 \le p \le \frac{n}{n-1}$, there exists $C = C(n, p, \Omega)$ s.t.

$$\|f\|_{L^p} \le C\,\|f\|_{BV} \tag{1.23}$$

Proof. For any $f \in BV(\Omega)$, by smooth approximation Theorem 1.1.2, choose $\{f_j\} \subset C^\infty(\Omega)$ s.t. $\|f_j - f\|_{L^1} \to 0$ and $\int_\Omega |Df| = \lim_{j\to 0} |Df_j|$. Then there exists M large enough s.t. $\|f_j\|_{BV} \le M$ uniformly. Since $C^\infty(\Omega) \subset W^{1,1}(\Omega)$, by Sobolev Embedding (1.19), for any $1 \le p \le \frac{n}{n-1}$, there exists $C = C(n, p, \Omega)$ s.t.

$$\|f_j\|_{L^p} \le C\left(\|f_j\|_{L^1} + \|Df_j\|_{L^1}\right) \le CM$$

uniformly in j. If $p = 1$, by definition of BV norm there's nothing to prove. For $1 < p \le \frac{n}{n-1}$, the uniform boundedness of f_j in L^p implies, from reflexivity of L^p and Banach Alaoglu, a weakly convergent subsequence in L^p. Still denoting f_j, ones has $f_0 \in L^p$ s.t. $f_j \rightharpoonup f_0$ in L^p. Since Ω is bounded, by Hölder, a priori one knows $f_j,\ f_0 \in L^1(\Omega)$, and for any $g \in (L^1(\Omega))^* = L^\infty(\Omega)$ (so $g^{\frac{p-1}{p}} \in L^{p'}(\Omega)$)

$$\left|\int_\Omega (f_j - f_0)\, g\right| dx = \left|\int_\Omega (f_j - f_0) g^{\frac{p-1}{p}}\, g^{\frac{1}{p}}\right| dx \le \left|\int_\Omega (f_j - f_0) g^{\frac{p-1}{p}}\, dx\right| \left\|g^{\frac{1}{p}}\right\|_{L^\infty(\Omega)} \to 0$$

hence one has $f_j \rightharpoonup f_0$ in L^1. But since we already know $f_j \to f$ in L^1, by uniqueness of L^1 strong limit, $f_0 = f$. Finally, by lower semicontinuity of weak convergence,

$$\|f\|_{L^p} \le \liminf_{j\to 0} \|f_j\|_{L^p} \le C \liminf_{j\to 0}\left(\|f_j\|_{L^1} + \|Df_j\|_{L^1}\right) = C\,\|f\|_{BV}$$

□

Theorem 1.1.4 (Compactness). *$\Omega \subset \mathbb{R}^n$ bounded open. $\partial\Omega$ Lipschitz continuous. Then*

$$BV(\Omega) \subset\subset L^p(\Omega) \qquad \forall\ 1 \le p < \frac{n}{n-1} \tag{1.24}$$

i.e., each uniformly bounded sequence $\{f_j\}$ in $BV(\Omega)$ norm has a convergent subsequence $\{f_{j_k}\}$ in $L^p(\Omega)$ norm for each $p \in [1, \frac{n}{n-1})$. Moreover, the limiting function $f \in BV(\Omega)$.

Proof. Let $\{f_j\} \subset BV(\Omega)$ uniformly bounded by $\|f_j\|_{BV(\Omega)} \leq M$. By smooth approximation Theorem 1.1.2, $\forall\, j$, choose $\tilde{f}_j \in C^\infty(\Omega)$ s.t.

$$\int_\Omega |f_j - \tilde{f}_j| < \frac{1}{j}, \qquad \int_\Omega |D\tilde{f}_j| dx \leq M + 2$$

Now since $\{\tilde{f}_j\} \subset C^\infty(\Omega)$ is uniformly bounded in $W^{1,1}(\Omega)$ norm, by Rellich (1.21), there exists convergent subsequence, still denoting $\tilde{f}_j$, in L^p for any $1 \leq p < \frac{n}{n-1}$. Fix any such p, let $f \in L^p(\Omega)$ s.t. $\left\|\tilde{f}_j - f\right\|_{L^p} \to 0$. Note Ω is bounded, hence Hölder inequality gives convergence in L^1 (p' Hölder conjugate w.r.t p)

$$\int_\Omega |f - \tilde{f}_j|\, dx \leq \left(\int_\Omega |f - \tilde{f}_j|^p\, dx\right)^{\frac{1}{p}} |\Omega|^{\frac{1}{p'}} \to 0$$

and then one may apply semicontinuity (1.7) which gives

$$\int_\Omega |Df| \leq \liminf_{j\to\infty} |D\tilde{f}_j| dx \leq M + 2 < \infty$$

to conclude $f \in BV(\Omega)$. It suffices to show $\|f_j - f\|_{L^p} \to 0$. But by Minkowski

$$\|f_j - f\|_{L^p} \leq \left\|f_j - \tilde{f}_j\right\|_{L^p} + \left\|\tilde{f}_j - f\right\|_{L^p}$$

where the former term convergence due to BV Embedding (1.22) and DCT

$$|f_j - \tilde{f}_j|^p \leq |f_j|^p + |\tilde{f}_j|^p \in L^1(\Omega) \implies \left\|f_j - \tilde{f}_j\right\|_{L^p} \to 0$$

and the latter term converges by Rellich (1.21)

□

Theorem 1.1.5 (Existence of Minimizing Caccioppoli Set)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded open, and let L be a Caccioppoli Set. Then there exists a Borel set E whose characteristic function φ_E minimizes the functional $\int |D\varphi_F|$ among all Borel sets F that agrees with L outside Ω, i.e., $\exists$ E Borel s.t. $E = L$ outside Ω and*

$$\int |D\varphi_E| \leq \int |D\varphi_F| \tag{1.25}$$

for any $F \subset \mathbb{R}^n$ Borel s.t. $F = L$ outside Ω.

Proof. One wish to use compactness that extracts a convergent subsequence in L^1. But notice we have no information about regularity of $\partial\Omega$, hence we first take $R > 0$ large s.t. $\Omega \subset\subset B_R(0)$ ball of radius R and we work with B_R. Take a minimizing sequence of sets $\{E_j\}$ s.t. $E_j = L$ outside Ω for any j and

$$\lim_{j\to\infty} \int_{B_R} |D\varphi_{E_j}| = \inf\{\int_{B_R} |D\varphi_F| \mid F = L \text{ outside } \Omega\} \tag{1.26}$$

notice L itself agrees with L outside Ω and since L is a Caccioppoli set, on B_R bounded open, $\int_{B_R} |D\varphi_L| < \infty$. Hence the RHS of (1.26) $< \infty$. Now we may take M large enough so $\int_{B_R} |D\varphi_{E_j}| < M$ uniformly bounded. And since B_R are bounded, $\varphi_{E_j} \in L^1(B_R)$ for any j, and in particular, $\left\|\varphi_{E_j}\right\|_{L^1(B_R)} \leq |B_R| < \infty$ uniformly, so $\{\varphi_{E_j}\} \subset BV(B_R)$ is uniformly bounded in BV norm. B_R has smooth boundary, so Theorem 1.1.4 gives a convergent subsequence $\varphi_{E_j} \to f$ in $L^1(B_R)$. Again passing to subsequence, $\varphi_{E_j} \to f$ pointwise a.e., but φ_{E_j} are characteristic functions, so $f = \varphi_E$ agrees with characteristic function of some Borel set E a.e. Indeed $E = E_j = L$ outside Ω. And since $\varphi_{E_j} \to \varphi_E$ in $L^1(B_R)$, by semicontinuity (1.7), $\int_{B_R} |D\varphi_E| \leq \lim_{j\to\infty} \int_{B_R} |D\varphi_{E_j}|$

$$\int_{B_R} |D\varphi_E| = \inf\{\int_{B_R} |D\varphi_F| \mid F = L \text{ outside } \Omega\}$$

Finally we recover estimate on $\mathbb{R}^n$ from B_R. For any $F \subset \mathbb{R}^n$ Borel s.t. $F = L$ outside Ω

$$\begin{aligned}
\int |D\varphi_E| &= \int_{B_R} |D\varphi_E| + \int_{B_R^c} |D\varphi_E| = \int_{B_R} |D\varphi_E| + \int_{B_R^c} |D\varphi_L| \\
&\leq \int_{B_R} |D\varphi_F| + \int_{B_R^c} |D\varphi_L| = \int_{B_R} |D\varphi_F| + \int_{B_R^c} |D\varphi_F| = \int |D\varphi_F|
\end{aligned}$$

□

Remark 1.1.6. *One has information for the minimizing set E from Theorem 1.1.5.*

- *L determines boundary values for E. Since $D\varphi_E$ is supported within ∂E, or more particularly, imagine E smooth so $\int_\Omega |D\varphi_E| = H_{n-1}(\partial E \cap \Omega)$ really measures the surface area of ∂E within Ω, then (1.25) indicates that '∂E within Ω' minimizes the surface area for all 'sets **within** Ω that has boundary $\partial L \cap \partial\Omega$'.*

- *Imagine $\partial L \cap \partial\Omega$ fixed, then it determines a surface spanning $\partial L \cap \partial\Omega$. But now curve the portion $\Omega \cap L$ towards Ω, it serves as obstacle forcing '∂E within Ω' away from the minimal surface spanned by $\partial L \cap \partial\Omega$.*

1.1.4 Coarea formula and Smooth Approximation of Caccioppolis sets

One shall recall Coarea formula for Lipschitz functions

Lemma 1.1.4 (Coarea Formula)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz for $n \geq m$. Then for any $A \subset \mathbb{R}^n$ Borel*

$$\int_A \sqrt{\det(Df^*Df)}(x)\, dx = \int_{\mathbb{R}^m} H_{n-m}\left(A \cap f^{-1}(y)\right)\, dy \tag{1.27}$$

With the Classical Coarea formula, one may prove for BV functions.

Theorem 1.1.6 (Coarea Formula)**.** *$\Omega \subset \mathbb{R}^n$ open. $f \in BV(\Omega)$. Denote $F_t := \{x \in \Omega \mid f(x) < t\}$, then*

$$\int_\Omega |Df| = \int_{-\infty}^{\infty} \left(\int_\Omega |D\varphi_{F_t}|\right)\, dt \tag{1.28}$$

Proof. $\leq$. First let $f \geq 0$. $\forall\, x \in \Omega$, $f(x) = \int_0^\infty \varphi_{F_t^c}\, dt = \int_0^\infty (1 - \varphi_{F_t})\, dt$, so $\forall\, g \in C_0^1(\Omega; \mathbb{R}^n)$ s.t. $|g| \leq 1$

$$\int_\Omega f \operatorname{div} g\, dx = \int_\Omega \left(\int_0^\infty (1 - \varphi_{F_t})\, dt\right) \operatorname{div} g\, dx = \int_0^\infty \left(\int_\Omega \operatorname{div} g\, dx - \int_\Omega \varphi_{F_t} \operatorname{div} g\, dx\right)\, dt$$

By Fubini, and then note compact support of g

$$= -\int_0^\infty \int_\Omega \varphi_{F_t} \operatorname{div} g\, dx\, dt \leq \int_0^\infty \int_\Omega |D\varphi_{F_t}|\, dt$$

Then let $f \leq 0$. $\forall\, x \in \Omega$, $f(x) = -\int_{-\infty}^0 \varphi_{F_t}\, dt$, so $\forall\, g \in C_0^1(\Omega; \mathbb{R}^n)$ s.t. $|g| \leq 1$

$$\int_\Omega f \operatorname{div} g\, dx = -\int_\Omega \left(\int_{-\infty}^0 \varphi_{F_t}\, dt\right) \operatorname{div} g\, dx = -\int_{-\infty}^0 \left(\int_\Omega \varphi_{F_t} \operatorname{div} g\, dx\right)\, dt \leq \int_{-\infty}^0 \int_\Omega |D\varphi_{F_t}|\, dt$$

Hence for any $f \in BV(\Omega)$, write $f = f^+ - f^-$ for $f^+,\, f^- \geq 0$, so

$$\int_\Omega f \operatorname{div} g\, dx \leq \int_\Omega \left(f^+ - f^-\right) \operatorname{div} g\, dx \leq \int_{-\infty}^\infty \int_\Omega |D\varphi_{F_t}|\, dt$$

taking supremum in g gives $\int_\Omega |Df| \leq \int_{-\infty}^\infty \int_\Omega |D\varphi_{F_t}|\, dt$.
$\geq$. One first show (1.28) for $f \in C(\Omega)$ continuous piecewise linear function. Let $\Omega = \bigcup_{i=1}^N \Omega_i$ for Ω_i disjoint, open where $f(x) = \langle a_i,\, x\rangle + b_i$ for $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, $x \in \Omega_i$. Then $\int_\Omega |Df| = \sum_{i=1}^N |a_i||\Omega_i|$. On the other hand, F_t now has piecewise smooth boundary, so

$$\int_{\Omega_i} |D\varphi_{F_t}| = H_{n-1}(\partial F_t \cap \Omega_i) = H_{n-1}\{x \in \Omega_i \mid f(x) = t\} = H_{n-1}\{x \in \Omega_i \mid \langle a_i,\, x\rangle + b_i = t\}$$

Hence integrating w.r.t. t and by change of coordinates

$$\begin{aligned}
\int_{-\infty}^\infty \int_{\Omega_i} |D\varphi_{F_t}|\, dt &= \int_{-\infty}^\infty H_{n-1}\{x \in \Omega_i \mid \langle a_i,\, x\rangle + b_i = t\}\, dt \\
&= \int_{-\infty}^\infty |a_i| H_{n-1}\left\{x \in \Omega_i \mid \frac{\langle a_i,\, x\rangle}{|a_i|} + \frac{b_i}{|a_i|} = \frac{t}{|a_i|}\right\}\, d\left(\frac{t}{|a_i|}\right) \\
&= |a_i| \int_{-\infty}^\infty H_{n-1}\left(\Omega_i \cap \left\{\frac{\langle a_i,\, x\rangle}{|a_i|} + \frac{b_i}{|a_i|} = t\right\}\right)\, dt
\end{aligned}$$

using Classical Coarea formula (1.27) with m = 1

$$= |a_i| \int_{\Omega_i} 1\, dx = |a_i||\Omega_i|$$

hence for $f \in C(\Omega)$ piecewise linear, (1.28) holds

$$\int_\Omega |Df| = \sum_{i=1}^N |a_i||\Omega_i| = \sum_{i=1}^N \int_{-\infty}^\infty \int_{\Omega_i} |D\varphi_{F_t}|\, dt = \int_{-\infty}^\infty \int_\Omega |D\varphi_{F_t}|\, dt$$

Now take any $f \in C^\infty(\Omega)$, approximate using sequence of $\{f_j\} \subset C(\Omega)$ continuous piecewise linear functions in $W^{1,1}(\Omega)$ norm. In particular, one has

$$\|f - f_j\|_{L^1(\Omega)} \to 0, \qquad \|Df\|_{L^1(\Omega)} = \lim_{j\to 0} \|Df_j\|_{L^1(\Omega)} \tag{1.29}$$

where the latter follows from $\|Df - Df_j\|_{L^1(\Omega)} \to 0$ and DCT. Denoting $F_{j,t} := \{x \in \Omega \mid f_j(x) < t\}$, one has

$$|f(x) - f_j(x)| = \int_{-\infty}^\infty |\varphi_{F_t}(x) - \varphi_{F_{j,t}}(x)|\, dt \implies \|f - f_j\|_{L^1(\Omega)} = \int_{-\infty}^\infty \int_\Omega |\varphi_{F_t}(x) - \varphi_{F_{j,t}}(x)|\, dx\, dt \to 0$$

hence there exists a subsequence $\varphi_{F_{j,t}} \to \varphi_{F_t}$ in $L^1(\Omega)$ a.e. t. Since (1.28) holds for each f_j,

$$\int_\Omega |Df| = \lim_{j\to 0} \int_\Omega |Df_j| = \lim_{j\to 0} \int_{-\infty}^\infty \int_\Omega |D\varphi_{F_{j,t}}|\, dt$$

one apply Fatou w.r.t. t

$$\geq \int_{-\infty}^\infty \left(\liminf_{j\to 0} \int_\Omega |D\varphi_{F_{j,t}}| \right) dt$$

then apply semicontinuity (1.7) for BV function

$$\geq \int_{-\infty}^\infty \int_\Omega |D\varphi_{F_t}|\, dt$$

and we conclude (1.28) for $f \in C^\infty(\Omega)$. But notice, we've really only used (1.29) in the above argument. Hence for any $f \in BV(\Omega)$, by Theorem 1.1.2, one may choose $\{f_j\} \subset C^\infty(\Omega)$ s.t. (1.29) holds. Then run the argument again, we conclude (1.28) for $f \in BV(\Omega)$.

□

To show for smooth approximation of sets, one needs Sard's lemma for smooth boundary construction.

Lemma 1.1.5 (Sard's Lemma). *$f : \mathbb{R}^n \to \mathbb{R}^m$ C^k where $k \geq \max\{n - m + 1, 1\}$. Let*

$$X := \{x \in \mathbb{R}^n \mid Jf(x) := \begin{bmatrix} \nabla f_1 \\ \cdots \\ \nabla f_m \end{bmatrix} (x) \text{ has rank } < m\}$$

denote the set of critical points of f. Then the image $f(X)$ has Lebesgue measure 0 in $\mathbb{R}^m$. In particular, if $m = 1$, then given C^k map $f : \mathbb{R}^n \to \mathbb{R}$ for $k \geq n$, one has

$$\partial\{x \in \mathbb{R}^n \mid f(x) < t\} = \{x \in \mathbb{R}^n \mid f(x) = t\} \quad C^k \text{ boundary for a.e. } t \in \mathbb{R} \tag{1.30}$$

Theorem 1.1.7 (Smooth approximation of Caccioppoli Set). *For $E \subset \mathbb{R}^n$ bounded Caccioppoli set, there exists E_j sets with C^∞ boundary s.t.*

$$\int |\varphi_{E_j} \to \varphi_E| dx \to 0 \qquad \int |D\varphi_E| = \lim_{j\to 0} \int |D\varphi_{E_j}| \tag{1.31}$$

Proof. Let η_ε be positive symmetric mollifier. For E Caccioppoli, one look at the mollification $(\varphi_E)_\varepsilon = \eta_\varepsilon * \varphi_E$. Since $(\varphi_E)_\varepsilon$ smooth and compactly supported, indeed $(\varphi_E)_\varepsilon \in BV(\mathbb{R}^n)$. Observe $0 \leq (\varphi_E)_\varepsilon \leq 1$ as inherited from φ_E, and denoting the set $E_{\varepsilon,t} := \{x \in \mathbb{R}^n \mid (\varphi_E)_\varepsilon(x) < t\}$, one has, by Coarea formula (1.28)

$$\int |D(\varphi_E)_\varepsilon| = \int_0^1 \left(\int |D\,\varphi_{E_{\varepsilon,t}}| \right) dt \tag{1.32}$$

But since E is bounded Caccioppoli, Corollary 1.1.1 gives $\varphi_E \in BV(\mathbb{R}^n)$. One may thus apply global mollification approximation (1.11)

$$\int |D\varphi_E| = \lim_{\varepsilon\to 0} \int |D(\varphi_E)_\varepsilon| = \lim_{\varepsilon\to 0} \int_0^1 \left(\int |D\,\varphi_{E_{\varepsilon,t}}| \right) dt$$

One now aims for the following claim. One wish to show for any $0 < t < 1$,

$$\int |\varphi_{E^c_{\varepsilon,t}} - \varphi_E|\, dx \le \frac{1}{\min\{1-t,\, t\}} \int |(\varphi_E)_\varepsilon - \varphi_E|\, dx \tag{1.33}$$

To do so, observe

$$\begin{aligned} (\varphi_E)_\varepsilon - \varphi_E \ge t \qquad & \textit{on } E^c_{\varepsilon,t} \setminus E \\ \varphi_E - (\varphi_E)_\varepsilon \ge 1-t \qquad & \textit{on } E \setminus E^c_{\varepsilon,t} \end{aligned}$$

Hence

$$\begin{aligned} \int |(\varphi_E)_\varepsilon - \varphi_E|\, dx &= \int_{E^c_{\varepsilon,t}\setminus E} |(\varphi_E)_\varepsilon - \varphi_E|\, dx + \int_{E\setminus E^c_{\varepsilon,t}} |(\varphi_E)_\varepsilon - \varphi_E|\, dx \\ &\ge t\, |E^c_{\varepsilon,t} \setminus E| + (1-t)\, |E \setminus E^c_{\varepsilon,t}| \ge \min\{1-t,\, t\} \int |\varphi_{E^c_{\varepsilon,t}} - \varphi_E|\, dx \end{aligned}$$

which gives (1.33). By mollification, since $\varphi_E \in L^1(\mathbb{R}^n) \subset BV(\mathbb{R}^n)$, $\|(\varphi_E)_\varepsilon - \varphi_E\|_{L^1} \to 0$, hence RHS of (1.33) converges to 0 as $\varepsilon \to 0$ for each t, implying $\left\|\varphi_{E^c_{\varepsilon,t}} - \varphi_E\right\|_{L^1} \to 0$ for each t. But since E bounded, $E^c_{\varepsilon,t} = \{x \mid (\varphi_E)_\varepsilon \ge t\}$ is also bounded for any $0 < t < 1$. And because $\partial E^c_{\varepsilon,t} = \{x \mid (\varphi_E)_\varepsilon = t\}$ is smooth, from example 1.1.2, one has $\varphi_{E^c_{\varepsilon,t}} \in BV(\mathbb{R}^n)$. Hence for $0 < t < 1$, one has semicontinuity (1.7)

$$\liminf_{\varepsilon\to 0} \int |D\varphi_{E^c_{\varepsilon,t}}| \ge \int |D\varphi_E|$$

But because $\mathrm{supp} D\varphi_{E^c_{\varepsilon,t}} \subset \partial E^c_{\varepsilon,t}$, under total variation, one has $\int |D\varphi_{E^c_{\varepsilon,t}}| = \int |D\varphi_{E_{\varepsilon,t}}|$. So

$$\int |D\varphi_E| = \lim_{\varepsilon\to 0} \int |D(\varphi_E)_\varepsilon| = \lim_{\varepsilon\to 0} \int_0^1 \left(\int |D\,\varphi_{E_{\varepsilon,t}}| \right) dt$$

By Fatou w.r.t. t

$$\ge \int_0^1 \left(\liminf_{\varepsilon\to 0} \int |D\varphi_{E_{\varepsilon,t}}| \right) dt = \int_0^1 \left(\liminf_{\varepsilon\to 0} \int |D\varphi^c_{E_{\varepsilon,t}}| \right) dt \ge \int |D\varphi_E|$$

now combining $\begin{cases} \liminf_{\varepsilon\to 0} \int |D\varphi_{E^c_{\varepsilon,t}}| \ge \int |D\varphi_E| \\ \int_0^1 \left(\liminf_{\varepsilon\to 0} \int |D\varphi_{E^c_{\varepsilon,t}}| \right) dt = \int |D\varphi_E| \end{cases}$ one must have for a.e. $0 < t < 1$

$$\liminf_{\varepsilon\to 0} \int |D\varphi_{E^c_{\varepsilon,t}}| = \int |D\varphi_E|$$

Now one is ready to apply Sard's lemma (1.30) to the set $\partial E^c_{\varepsilon,t} = \{x \in \mathbb{R}^n \mid (\varphi_E)_\varepsilon = t\}$, resulting in smooth boundary of $\partial E^c_{\varepsilon,t}$ for a.e. $0 < t < 1$. Take one such t. we have obtained

$$\begin{cases} \partial E^c_{\varepsilon,t} \quad \textit{smooth} \\ \left\|\varphi_{E^c_{\varepsilon,t}} - \varphi_E\right\|_{L^1} \to 0 \\ \liminf_{\varepsilon\to 0} \int |D\varphi_{E^c_{\varepsilon,t}}| = \int |D\varphi_E| \end{cases}$$

Take subsequence ε_j s.t. $\varepsilon_j \to 0$ as $j \to \infty$ and $\int |D\varphi_E| = \lim_{j\to 0} \int |D\varphi^c_{E_{\varepsilon_j,t}}|$. Define $E_j := E^c_{\varepsilon_j,t}$. □

Remark 1.1.7. *Notice E_j bounded and smooth ensures $\varphi_{E_j} \subset BV(\mathbb{R}^n)$, and E bounded Caccioppoli ensures $\varphi_E \in BV(\mathbb{R}^n)$. Hence one may apply (1.8), so that for any $A \subset \mathbb{R}^n$ open*

$$\int_A |D\varphi_E| = \lim_{j\to 0} \int_A |D\varphi_{E_j}|$$

1.1.5 Isoperimetric Inequality

One shall first recall from Sobolev Space the GNS inequality as the tool from (1.19) and Poincaré Lemma

Lemma 1.1.6 (GNS Inequality)**.** *$1 \le p < n$. Then there exists $C = C(n,\, p)$ s.t.*

$$\|f\|_{L^{\frac{np}{n-p}}(\mathbb{R}^n)} \le C\, \|Df\|_{L^p(\mathbb{R}^n)} \qquad \forall\, f \in C^1_0(\mathbb{R}^n) \tag{1.34}$$

Lemma 1.1.7 (Poincaré). *$\Omega \subset \mathbb{R}^n$ open, bounded, connected. $\partial\Omega$ Lipschitz continuous. $1 \le p \le \infty$. There there exists $C = C(n, p, \Omega)$ s.t.*

$$\left\| f - \fint_\Omega f\,dy \right\|_{L^p(\Omega)} \le C\,\|Df\|_{L^p(\Omega)} \qquad \forall\, f \in W^{1,p}(\Omega) \tag{1.35}$$

Corollary 1.1.2. *There exists $C_1 = C_1(n)$ and $C_2 = C_2(n)$ s.t.*

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le C_1\,\|Df\|_{L^1(\mathbb{R}^n)} \qquad \forall\, f \in C_0^\infty(\mathbb{R}^n) \tag{1.36}$$

$$\|f - f_\rho\|_{L^{\frac{n}{n-1}}(B_\rho)} \le C_2\,\|Df\|_{L^1(B_\rho)} \qquad \forall\, f \in C^\infty(B_\rho) \tag{1.37}$$

where $f_\rho := \fint_{B_\rho} f\,dy = \frac{1}{|B_\rho|}\int_{B_\rho} f\,dy$.

Proof. Apply (1.34) with $p = 1$ yields (1.36). Apply (1.19) with $\Omega = B_\rho$, $p = 1$ and $q = \frac{n}{n-1}$ gives

$$\|f - f_\rho\|_{L^{\frac{n}{n-1}}(B_\rho)} \le C\,\|f - f_\rho\|_{W^{1,1}(B_\rho)} = C\left(\|f - f_\rho\|_{L^1(B_\rho)} + \|Df\|_{L^1(B_\rho)}\right) \le C_2\,\|Df\|_{L^1(B_\rho)}$$

where the last inequality uses (1.35). □

One immediately has Sobolev Inequalities for BV function.

Theorem 1.1.8 (Sobolev for BV). *There exists $C_1 = C_1(n)$ and $C_2 = C_2(n)$ s.t.*

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le C_1 \int |Df| \qquad \forall\, f \in BV(\mathbb{R}^n) \text{ and } \operatorname{supp} f \text{ compact} \tag{1.38}$$

$$\|f - f_\rho\|_{L^{\frac{n}{n-1}}(B_\rho)} \le C_2 \int_{B_\rho} |Df| \qquad \forall\, f \in BV(B_\rho) \tag{1.39}$$

where $f_\rho := \fint_{B_\rho} f\,dy = \frac{1}{|B_\rho|}\int_{B_\rho} f\,dy$.

Proof. One mimic the proof in (1.23). For $f \in BV(\mathbb{R}^n)$ with $\operatorname{supp} f$ compact, by smooth approximation Theorem 1.1.2, there exists $\{f_j\} \subset C_0^\infty(\mathbb{R}^n)$ with uniform compact support s.t. $\|f_j - f\|_{L^1(\mathbb{R}^n)} \to 0$ and $\int |Df| = \lim\limits_{j\to\infty} \int |Df_j|dx$. Now Df_j is uniformly bounded in L^1 on $\mathbb{R}^n$, say by M. So one has from (1.36), $\|f_j\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le C_1\,\|Df_j\|_{L^1(\mathbb{R}^n)} \le C_1 M$ uniformly bounded. Since $L^{\frac{n}{n-1}}$ is Reflexive, a uniformly bounded sequence in $L^{\frac{n}{n-1}}$ has a weakly convergent subsequence by Banach Alaoglu, say $f_j \rightharpoonup f_0$ in $L^{\frac{n}{n-1}}$. But with uniform compact support for f_j and f_0, one has $f_j \rightharpoonup f_0$ in L^1 by Hölder. Since we already know $f_j \to f$ in L^1, $f_0 = f$. Now by lower semicontinuity of weak convergence

$$\left(\int |f|^{\frac{n}{n-1}}\,dx\right)^{\frac{n-1}{n}} \le \lim_{j\to\infty}\left(\int |f_j|^{\frac{n}{n-1}}\,dx\right)^{\frac{n-1}{n}} \le C_1 \lim_{j\to\infty} \|Df_j\|_{L^1(\mathbb{R}^n)} = C_1 \int |Df|$$

thus we've proved (1.38). For $f \in BV(B_\rho)$, by smooth approximation Theorem 1.1.2, there exists $\{f_j\} \subset C^\infty(B_\rho)$ s.t. $\|f_j - f\|_{L^1(B_\rho)} \to 0$ and $\int_{B_\rho} |Df| = \lim\limits_{j\to\infty} \int_{B_\rho} |Df_j|dx$, so $\|Df_j\|_{L^1(B_\rho)}$ is uniformly bounded, and by (1.37), $\{f_j - (f_j)_\rho\}$ is uniformly bounded in $L^{\frac{n}{n-1}}(B_\rho)$. Hence there exists weakly convergent subsequence $f_j - (f_j)_\rho \rightharpoonup f_0$ in $L^{\frac{n}{n-1}}(B_\rho)$, thus since B_ρ bounded, $f_j - (f_j)_\rho \rightharpoonup f_0$ weakly in $L^1(B_\rho)$ via Hölder. But $f_j - (f_j)_\rho \to f - f_\rho$ in L^1, so $f - f_\rho = f_0$. Again by the lower semicontinuity one has (1.39)

$$\left(\int_{B_\rho} |f - f_\rho|^{\frac{n}{n-1}}\,dx\right)^{\frac{n-1}{n}} \le \lim_{j\to\infty}\left(\int_{B_\rho} |f_j - (f_j)_\rho|^{\frac{n}{n-1}}\,dx\right)^{\frac{n-1}{n}} \le C_2 \lim_{j\to\infty} \|Df_j\|_{L^1(B_\rho)} = C_2 \int_{B_\rho} |Df|$$

□

Theorem 1.1.9 (Isoperimetric Inequality). *For $E \subset \mathbb{R}^n$ bounded Caccioppoli, there exists $C_1 = C_1(n)$ and $C_2 = C_2(n)$ s.t. for any open ball $B_\rho \subset \mathbb{R}^n$ with radius ρ*

$$|E|^{\frac{n-1}{n}} \le C_1 \int |D\varphi_E| \tag{1.40}$$

$$\min\{|E \cap B_\rho|,\, |E^c \cap B_\rho|\}^{\frac{n-1}{n}} \le C_2 \int_{B_\rho} |D\varphi_E| \tag{1.41}$$

Proof. Since E bounded Caccioppoli, $\varphi_E \in BV(\mathbb{R}^n)$ and supp$\varphi_E = \overline{E}$ is compact, one apply (1.38) and so (1.40) holds. Now let $f = \varphi_E$, then $f_\rho = \frac{1}{|B_\rho|}\int_{B_\rho}\varphi_E = \frac{|E\cap B_\rho|}{|B_\rho|}$, so

$$\begin{aligned}\int_{B_\rho}|f-f_\rho|^{\frac{n}{n-1}}\,dx &= \int_{B_\rho\cap E}|1-f_\rho|^{\frac{n}{n-1}}\,dx + \int_{B_\rho\cap E^c}|f_\rho|^{\frac{n}{n-1}}\,dx\\ &= |B_\rho\cap E|\left(\frac{|E^c\cap B_\rho|}{|B_\rho|}\right)^{\frac{n}{n-1}} + |B_\rho\cap E^c|\left(\frac{|E\cap B_\rho|}{|B_\rho|}\right)^{\frac{n}{n-1}}\\ &\geq \min\{|B_\rho\cap E|,\,|B_\rho\cap E^c|\}\left(\left(1-\frac{|E\cap B_\rho|}{|B_\rho|}\right)^{\frac{n}{n-1}} + \left(\frac{|E\cap B_\rho|}{|B_\rho|}\right)^{\frac{n}{n-1}}\right)\end{aligned}$$

Hence taking $\frac{n-1}{n}$ power gives

$$\left(\int_{B_\rho}|f-f_\rho|^{\frac{n}{n-1}}\,dx\right)^{\frac{n-1}{n}} \geq \min\{|B_\rho\cap E|,\,|B_\rho\cap E^c|\}^{\frac{n-1}{n}}\left(\left(1-\frac{|E\cap B_\rho|}{|B_\rho|}\right)^{\frac{n}{n-1}} + \left(\frac{|E\cap B_\rho|}{|B_\rho|}\right)^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}$$

Notice for any $\theta \geq 1$ and $a,\,b \geq 0$, one has elementary inequality $(a+b)^\theta \leq 2^\theta\left(a^\theta + b^\theta\right)$. Letting $\theta = \frac{n}{n-1}$, $a = 1 - \frac{|E\cap B_\rho|}{|B_\rho|}$ and $b = \frac{|E\cap B_\rho|}{|B_\rho|}$, so

$$\left(\left(1-\frac{|E\cap B_\rho|}{|B_\rho|}\right)^{\frac{n}{n-1}} + \left(\frac{|E\cap B_\rho|}{|B_\rho|}\right)^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \geq \left(2^{\frac{-n}{n-1}}\cdot 1\right)^{\frac{n-1}{n}} = \frac{1}{2}$$

independent of size of B_ρ. Hence apply (1.39) one has (1.41).

□

1.2 Traces of BV Function

1.2.1 preliminary lemmas

Lemma 1.2.1 (Lebesgue Differentiation)**.** *$f \in L^1(\mathbb{R}^n)$. Then for a.e. $x \in \mathbb{R}^n$*

$$\lim_{\rho\to 0}\frac{1}{\rho^n}\int_{B_\rho}|f(x+y)-f(x)|\,dy = 0 \tag{1.42}$$

One need Zorn's lemma for a Covering argument.

Lemma 1.2.2 (Zorn's Lemma)**.** *One needs a few definitions to make sense of Zorn's lemma.*

- *A set P is partially ordered by $\leq$ if*
  1. *$\leq$ is reflexive: $x \leq x$ for any $x \in P$*
  2. *$\leq$ is anti-symmetric: $x \leq y$ and $y \leq x$ implies $x = y$*
  3. *$\leq$ is transitive: $x \leq y$ and $y \leq z$ implies $x \leq z$*

  Note not all elements in P are required to be comparable. If a subset $S \subset P$ that inherits the partial order $\leq$ has every pair of elements comparable, S is called totally ordered.

- *An element $m \in P$ with partial order $\leq$ is maximal if there does not exist $s \in P$ s.t. $s \neq m$ and $m \leq s$. Note 'maximal' here does not need m to be comparable with all other elements in P.*

- *Given subset $S \subset P$ that inherits the partial order $\leq$. An element $u \in P$ is an upper bound of S if for any $s \in S$, $s \leq u$.*

Zorn's Lemma claims: Given a nonempty partially order set $(P, \leq)$. If every nonempty subset $S \subset P$ that inherits the order $\leq$ and is totally bounded has an upper bound $u \in P$, then P contains at least one maximal element m with order $\leq$.

Lemma 1.2.3 (Covering Lemma)**.** *$A \subset \mathbb{R}^n$. $\rho : A \to (0,1)$. Then there exists countable set $\{x_i\} \subset A$ s.t.*

$$B_{\rho(x_i)}(x_i) \cap B_{\rho(x_j)}(x_j) = \varnothing \quad for\ i \neq j \tag{1.43}$$

$$A \subset \bigcup_{i=1}^{\infty} B_{3\rho(x_i)}(x_i) \tag{1.44}$$

Proof. For $k \geq 1$, let $A_k := \{x \in A \mid \frac{1}{2^k} \leq \rho(x) < \frac{1}{2^{k-1}}\}$. One wish to define a sequence of sets L_k for each k. If $A_k = \varnothing$, let $L_k := \varnothing$. WLOG, assume $A_1 \neq \varnothing$. Let $\mathcal{L}_1 := \{L \subset A_1 \mid \forall\, x,\, y \in L,\, x \neq y,\, B_{\rho(x)}(x) \cap B_{\rho(y)}(y) = \varnothing\}$. For nonempty A_1, $\mathcal{L}_1$ is indeed nonempty because both the empty set and singletons are elements of $\mathcal{L}_1$. Now order $\mathcal{L}_1$ with inclusion. For any subcollection of $\mathcal{L}_1$ totally ordered with inclusion, indeed their union is element of $\mathcal{L}_1$ and is upper bounded. Hence $\mathcal{L}_1$ contains a maximal element via Zorn's lemma, call it L_1. Now assume for $L_1, \cdots, L_k$, one obtain L_{k+1} via taking the maximal element of the following collection ordered with inclusion

$$\mathcal{L}_{k+1} := \{L \subset A_{k+1} \mid \forall\, x,\, y \in L_1 \cup L_2 \cup \cdots \cup L_k \cup L,\, x \neq y,\, B_{\rho(x)}(x) \cap B_{\rho(y)}(y) = \varnothing\}$$

Notice $\varnothing \in \mathcal{L}_{k+1}$ is always true so Zorn's lemma applies. L_{k+1} could be empty even if A_{k+1} is nonempty. Moreover, for each L_k, for any $M \subset \mathbb{R}^n$ compact, $M \cap L_k$ must contain finitely many elements otherwise $\{B_{\rho(x)}(x)\}_{x \in M \cap L_k}$ as open cover of $M \cap \overline{L}_k$ does not have finite subcover, contradicting compactness of $M \cap \overline{L}_k$. Hence let M truncate collections of balls $\{\overline{B}_j\}$ with radius $j \in \mathbb{N}$, so each $\overline{B}_j \cap L_k$ is finite for any j. Thus pass j to ∞, L_k is countable. So $L := \bigcup_{k=1}^{\infty} L_k$ is countable set satisfying (1.43). To see (1.44), take any $z \in A = \bigcup_{k=1}^{\infty} A_k$. There must exist k s.t. $z \in A_k$. Now since L_k is maximal element of $\mathcal{L}_k$, $L_k \cup \{z\} \notin \mathcal{L}_k$. Hence there must exist $x \in L_1 \cup \cdots \cup L_k$ s.t. $x \neq z$ and $B_{\rho(x)}(x) \cap B_{\rho(z)}(z) \neq \varnothing$. Note by definition of A_k, $\frac{1}{2^k} \leq \rho(z) < \frac{1}{2^{k-1}}$, and by definition of $L_1 \cup \cdots \cup L_k$, $\frac{1}{2^k} \leq \rho(x) < 1$. Hence $\frac{1}{2}\rho(z) < \rho(x)$. But the balls $B_{\rho(x)}(x) \cap B_{\rho(z)}(z) \neq \varnothing$, so $z \in B_{3\rho(x)}(x)$. □

Using the covering lemma, one obtains a boundary differentiation lemma analogous to Lemma 1.2.1.

- $B_r(x) := \{z \in \mathbb{R}^n \mid |x - z| < r\}$ ball with center x radius r in $\mathbb{R}^n$
- $\mathscr{B}_\rho(y) := \{t \in \mathbb{R}^{n-1} \mid |y - t| < \rho\}$ ball with center y radius ρ in $\mathbb{R}^{n-1}$
- Let $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_n > 0\}$, $y \in \mathbb{R}^{n-1} = \partial\mathbb{R}^n_+$, $\rho > 0$. Upper cylinder with center y radius and height ρ

$$C^+_\rho(y) := \{(z, t) \in \mathbb{R}^{n-1} \times (0, \infty) \mid |y - z| < \rho,\, 0 < t < \rho\} = \mathscr{B}_\rho(y) \times (0,\, \rho)$$

Lemma 1.2.4. *μ positive Radon measure on $\mathbb{R}^n_+$ with $\mu(\mathbb{R}^n_+) < \infty$. Then for H_{n-1}-a.e. $y \in \mathbb{R}^{n-1} = \partial\mathbb{R}^n_+$,*

$$\lim_{\rho \to 0} \frac{1}{\rho^{n-1}} \mu(C^+_\rho(y)) = 0 \tag{1.45}$$

Proof. It suffices to show $\forall\, k > 0$, $A_k := \{y \in \mathbb{R}^{n-1} \mid \limsup_{\rho \to 0} \frac{1}{\rho^{n-1}} \mu(C^+_\rho(y)) > \frac{1}{k}\}$ is of H_{n-1} measure zero. Given $\varepsilon > 0$. Note for any $y \in A_k$, there exists $\rho_y < \varepsilon$ s.t.

$$\frac{1}{\rho_y^{n-1}} \mu(C^+_{\rho_y}(y)) > \frac{1}{2k} \iff \rho_y^{n-1} < 2k\, \mu(C^+_{\rho_y}(y))$$

Choose $\{y_j\} \subset A_k$ as in Lemma 1.2.3 with $\rho(y_j) = \rho_{y_j}$ so that $\mathscr{B}_{\rho_{y_j}}(y_j)$ are disjoint and $A_k \subset \bigcup_{j=1}^{\infty} \mathscr{B}_{3\rho_{y_j}}(y_j)$.

$$H_{n-1}(A_k) \leq \sum_{j=1}^{\infty} H_{n-1}(\mathscr{B}_{3\rho_{y_j}}(y_j)) = \omega_{n-1} \sum_{j=1}^{\infty} (3\rho_{y_j})^{n-1} < \omega_{n-1} 3^{n-1} 2k \sum_{j=1}^{\infty} \mu(C^+_{\rho_{y_j}}(y_j))$$

But $C^+_{\rho_{y_j}}(y_j) = \mathscr{B}_{\rho_{y_j}}(y_j) \times (0,\, \rho_{y_j})$ are disjoint, and since $\rho_{y_j} < \varepsilon$ uniformly in j

$$H_{n-1}(A_k) \leq \omega_{n-1} 3^{n-1} 2k\, \mu\{x \in \mathbb{R}^n_+ \mid 0 < x_n < \varepsilon\}$$

for any $\varepsilon > 0$. But $\mu(\mathbb{R}^n_+) < \infty$, so $\mu\{x \in \mathbb{R}^n_+ \mid 0 < x_n < \varepsilon\} \to 0$ as $\varepsilon \to 0$, hence $H_{n-1}(A_k) = 0\ \forall\, k > 0$. □

1.2.2 Existence and Property of Trace on C_R

One first work with upper cylinder $C^+_R := C^+_R(0) = \mathscr{B}_R \times (0, R)$. Also denote $C_R := \mathscr{B}_R \times (-R,\, R)$.

Theorem 1.2.1 (Construction of Trace)**.** *$f \in BV(C^+_R)$. There exists $f^+ \in L^1(\mathscr{B}_R)$ s.t. for H_{n-1}-a.e. $y \in \mathscr{B}_R$*

$$\lim_{\rho \to 0} \frac{1}{\rho^n} \int_{C^+_\rho(y)} |f(z) - f^+(y)|\, dz = 0 \tag{1.46}$$

and for any $g \in C^1_0(C_R; \mathbb{R}^n)$, one has

$$\int_{C^+_R} f \operatorname{div} g\, dx = -\int_{C^+_R} \langle g,\, Df\rangle - \int_{\mathscr{B}_R} f^+ g_n\, dH_{n-1} \tag{1.47}$$

Definition 1.2.1 (Trace of BV Function)**.** *$f \in BV(C_R^+)$. $f^+ \in L^1(\mathscr{B}_R)$ in Theorem 1.2.1 is trace of f on $\mathscr{B}_R$. Indeed (1.46) implies for H_{n-1}-a.e. $y \in \mathscr{B}_R$*

$$f^+(y) = \lim_{\rho \to 0} \frac{1}{|C_\rho^+(y)|} \int_{C_\rho^+(y)} f(z)\, dz \tag{1.48}$$

Proof. First suppose $f \in C^\infty(C_R^+)$. Then for any $0 < \varepsilon < R$, define $f^\varepsilon : \mathscr{B}_R \to \mathbb{R}$ as $f^\varepsilon(y) := f(y,\, \varepsilon)$. Hence denoting $Q_{\varepsilon',\varepsilon} := \mathscr{B}_R \times (\varepsilon',\, \varepsilon)$ for $0 \le \varepsilon' < \varepsilon \le R$, one has from FTC

$$\int_{\mathscr{B}_R} |f^\varepsilon(y) - f^{\varepsilon'}(y)|\, dH_{n-1}(y) \le \int_{\mathscr{B}_R} \int_{\varepsilon'}^{\varepsilon} |D_n f(y,t)|\, dt\, dH_{n-1}(y) = \int_{Q_{\varepsilon',\varepsilon}} |D_n f|\, dx \tag{1.49}$$

Since f smooth, RHS Cauchy in ε gives LHS Cauchy in ε, thus $\exists\, f^+ \in L^1(\mathscr{B}_R)$ s.t. $\|f^\varepsilon - f^+\|_{L^1(\mathscr{B}_R)} \to 0$. Take any $g \in C_0^1(C_R;\mathbb{R}^n)$, Since f smooth, for any $0 < \varepsilon < R$, and let $\nu = (\nu^1, \cdots, \nu^n)$ denote unit normal w.r.t. $\mathscr{B}_R \times \{x_n = \varepsilon\}$ and pointing downwards to $\mathbb{R}^{n-1}$, i.e., $\nu = (0, \cdots, 0, -1)$

$$\begin{aligned}
\int_{Q_{\varepsilon,R}} f \operatorname{div} g\, dx &= -\int_{Q_{\varepsilon,R}} \langle g,\, Df\rangle + \int_{\mathscr{B}_R \times \{x_n = \varepsilon\}} f(y,\, \varepsilon)\, g(y,\, \varepsilon) \cdot \nu\, dH_{n-1}(y) \\
&= -\int_{Q_{\varepsilon,R}} \langle g,\, Df\rangle - \int_{\mathscr{B}_R \times \{x_n = \varepsilon\}} f(y,\, \varepsilon)\, g_n(y,\, \varepsilon)\, dH_{n-1}(y) \\
&= -\int_{Q_{\varepsilon,R}} \langle g,\, Df\rangle - \int_{\mathscr{B}_R} f^\varepsilon(y)\, g_n^\varepsilon(y)\, dH_{n-1}(y)
\end{aligned}$$

letting $\varepsilon \to 0$, one obtain (1.47) for f smooth. To see for (1.46), for any $y \in \mathscr{B}_R$ and $0 < \rho < R$ s.t. $C_\rho^+(y) \subset C_R^+$

$$\begin{aligned}
\int_{C_\rho^+(y)} |f(z) - f^+(y)|\, dz &= \int_{\mathscr{B}_\rho(y)} \int_0^\rho |f(\eta,\, t) - f^+(y)|\, dt\, dH_{n-1}(\eta) \\
&\le \int_{\mathscr{B}_\rho(y)} \int_0^\rho |f(\eta,\, t) - f^+(\eta)|\, dt\, dH_{n-1}(\eta) + \int_{\mathscr{B}_\rho(y)} \int_0^\rho |f^+(\eta) - f^+(y)|\, dt\, dH_{n-1}(\eta) \\
&= \int_{\mathscr{B}_\rho(y)} \int_0^\rho |f(\eta,\, t) - f^+(\eta)|\, dt\, dH_{n-1}(\eta) + \rho \int_{\mathscr{B}_\rho(y)} |f^+(\eta) - f^+(y)|\, dH_{n-1}(\eta)
\end{aligned}$$

notice upon multiplying by ρ^{-n}, the second term goes to 0 for H_{n-1}-a.e. y due to Lebesgue Differentiation 1.2.1. For the first term, use Fubini and mimic (1.49)

$$\begin{aligned}
\int_{\mathscr{B}_\rho(y)} \int_0^\rho |f(\eta,\, t) - f^+(\eta)|\, dt\, dH_{n-1}(\eta) &= \int_0^\rho \int_{\mathscr{B}_\rho(y)} |f^t(\eta) - f^+(\eta)|\, dH_{n-1}(\eta)\, dt \\
&\le \int_0^\rho \int_{\mathscr{B}_\rho(y)} \int_0^t |D_n f(\eta,\, \xi)|\, d\xi\, dH_{n-1}(\eta)\, dt \\
&\le \int_0^\rho \int_{Q_{0,t}(y)} |Df|\, dx\, dt \le \rho \int_{C_\rho^+(y)} |Df|
\end{aligned}$$

now multiplying by ρ^{-n} and notice $|Df|$ is Radon measure on C_R^+ that is finite, one may use (1.45) with $\mu = |Df|$. Hence for H_{n-1}-a.e. $y \in \mathscr{B}_R$

$$\frac{1}{\rho^n} \int_{C_\rho^+(y)} |f(z) - f^+(y)|\, dz \le \frac{1}{\rho^{n-1}} \int_{C_\rho^+(y)} |Df| + \frac{1}{\rho^{n-1}} \int_{\mathscr{B}_\rho(y)} |f^+(\eta) - f^+(y)|\, dH_{n-1}(\eta) \to 0$$

and one concludes (1.46) for f smooth. In general for $f \in BV(C_R^+)$, approximate using $\{f_j\} \subset C^\infty(C_R^+)$ via Theorem 1.1.2. Recall remark (1.18), for any j, given n and H_{n-1}-a.e. $y \in \mathscr{B}_R$

$$\lim_{\rho \to 0} \frac{1}{\rho^n} \int_{C_\rho^+(y)} |f(z) - f_j(z)|\, dz = 0$$

Hence combining with f_j satisfying (1.46)

$$\frac{1}{\rho^n} \int_{C_\rho^+(y)} |f(z) - f_j^+(y)|\, dz \le \frac{1}{\rho^n} \int_{C_\rho^+(y)} |f(z) - f_j(z)|\, dz + \frac{1}{\rho^n} \int_{C_\rho^+(y)} |f_j(z) - f_j^+(y)|\, dz \to 0$$

for any j. Thus by uniqueness of L^1 limit, all traces f_j^+ coincide H_{n-1}-a.e. $y \in \mathscr{B}_R$. So define $f^+ := f_j^+$ for any such trace. One has (1.46) for $f \in BV(C_R^+)$. Finally, since $\|f - f_j\|_{L^1(C_R^+)} \to 0$ and $\int_{C_R^+} |Df_j| \to \int_{C_R^+} |Df|$, one wish to deduce (1.47) from

$$\int_{C_R^+} f_j \operatorname{div} g\, dx = -\int_{C_R^+} \langle g,\, Df_j\rangle - \int_{\mathscr{B}_R} f_j^+\, g_n\, dH_{n-1}$$

The first term converges due to $\|f - f_j\|_{L^1(C_R^+)} \to 0$ and the last term does not need to converge as $f^+ = f_j^+$ for any j. For the second term, note $\int_{C_R^+} |Df_j| \to \int_{C_R^+} |Df|$ convergence ensures uniform boundedness of $\int_{C_R^+} |Df_j|$. By Banach Alaoglu, the closed unit ball in norm is compact in the weak* topology. Hence identifying $\int_{C_R^+} |Df|$ as norm, there exists subsequence s.t. $Df_j \overset{*}{\rightharpoonup} Df$. But the vague topology convergence $\int_{C_R^+} \langle g, Df_j \rangle \to \int_{C_R^+} \langle g, Df \rangle$ is essentially the weak* topology convergence. Hence we're done.

□

Proposition 1.2.1 (Approximation in BV implies Approximation in Trace)**.** *$f \in BV(C_R^+)$. If $\{f_j\} \subset BV(C_R^+)$ s.t. $f_j \to f$ in $L^1(C_R^+)$ and*

$$\lim_{j\to\infty} \int_{C_R^+} |Df_j| = \int_{C_R^+} |Df| \tag{1.50}$$

then

$$\lim_{j\to\infty} \int_{\mathscr{B}_R} |f_j^+ - f^+| \, dH_{n-1}(y) = 0 \tag{1.51}$$

Proof. For any $0 < \beta < R$, consider $Q_{0,\beta} := \mathscr{B}_R \times (0, \beta)$. Define $f_\beta : \mathscr{B}_R \to \mathbb{R}$ s.t. $f_\beta(y) := \frac{1}{\beta} \int_0^\beta f(y,t)\, dt$ for any $f \in BV(C_R^+)$. Then for a.e. β

$$\begin{aligned}\int_{\mathscr{B}_R} |f^+(y) - f_\beta(y)| \, dH_{n-1}(y) &= \int_{\mathscr{B}_R} |f^+(y) - \frac{1}{\beta} \int_0^\beta f(y,t)\, dt| \, dH_{n-1}(y) \\ &= \frac{1}{\beta} \int_0^\beta \int_{\mathscr{B}_R} |f^+(y) - f(y,t)| \, dH_{n-1}(y)\, dt \ \leq \frac{1}{\beta} \int_0^\beta \int_{Q_{0,t}} |Df| \, dx\, dt \leq \int_{Q_{0,\beta}} |Df| \, dx\end{aligned} \tag{1.52}$$

where the last line uses (1.49), initially shown for smooth f. To make sense of (1.49) for $f \in BV(C_R^+)$, one precisely needs smooth approximation from Theorem 1.1.2 where $\|f_\varepsilon \to f\|_{L^1(C_R^+)}$ implies for a.e. t

$$\int_{\mathscr{B}_R} |f_\varepsilon^+(y) - f_\varepsilon(y,t)| \, dH_{n-1}(y) \ \to \ \int_{\mathscr{B}_R} |f^+(y) - f(y,t)| \, dH_{n-1}(y)$$

and $\int_{C_R^+} |Df_\varepsilon| \to \int_{C_R^+} |Df|$ implies via (1.8) ($\int_{\mathscr{B}_R \times \{t\}} |Df| = 0$ for a.e. t otherwise uncountably many disjoint summing up contradicts $f \in BV(C_R^+)$) that $\int_{Q_{0,t}} |Df_\varepsilon| \to \int_{Q_{0,t}} |Df|$. Hence for $\{f_j\} \subset BV(C_R^+)$ as assumed

$$\int_{\mathscr{B}_R} |f_j^+ - f^+| \, dH_{n-1}(y) \leq \int_{\mathscr{B}_R} |f_j^+ - (f_j)_\beta| \, dH_{n-1}(y) + \int_{\mathscr{B}_R} |(f_j)_\beta - f_\beta| \, dH_{n-1}(y) + \int_{\mathscr{B}_R} |f_\beta - f^+| \, dH_{n-1}(y)$$

using (1.52)

$$\leq \int_{Q_{0,\beta}} |Df_j| + \int_{\mathscr{B}_R} |(f_j)_\beta - f_\beta| \, dH_{n-1}(y) + \int_{Q_{0,\beta}} |Df|$$

the middle term writes, using $\|f_j - f\|_{L^1(C_R^+)} \to 0$

$$\int_{\mathscr{B}_R} |(f_j)_\beta - f_\beta| \, dH_{n-1}(y) = \frac{1}{\beta} \int_0^\beta \int_{\mathscr{B}_R} |f_j(y,t) - f(y,t)| \, dH_{n-1}(y)\, dt = \frac{1}{\beta} \int_{C_R^+} |f_j - f| \, dx \to 0$$

Thus, since for a.e. β, $\int_{Q_{0,\beta}} |Df_j| \to \int_{Q_{0,\beta}} |Df|$, one has

$$\limsup_{j\to\infty} \int_{\mathscr{B}_R} |f_j^+ - f^+| \, dH_{n-1}(y) \leq 2 \int_{Q_{0,\beta}} |Df|$$

for a.e. β. Thus using $f \in BV(C_R^+)$ so $\int_{Q_{0,\beta}} |Df| \to 0$ as $\beta \to 0$, one arrives at (1.51). □

Note for $C_R^- := \mathscr{B}_R \times (-R, 0)$, one may similarly define $f^- \in L^1(\mathscr{B}_R)$ as trace for the function $f \in BV(C_R^-)$ via Theorem 1.2.1.

Proposition 1.2.2 (Extension Property for BV)**.** *For $f_1 \in BV(C_R^+)$ and $f_2 \in BV(C_R^-)$, let $f^+, f^- \in L^1(\mathscr{B}_R)$ be their trace respectively. Then for $f : C_R = \mathscr{B}_R \times (-R, R) \to \mathbb{R}$ defined as $f := \begin{cases} f_1 & \text{in } C_R^+ \\ f_2 & \text{in } C_R^- \end{cases}$, one has $f \in BV(C_R)$ and*

$$\int_{\mathscr{B}_R} |f^+ - f^-| \, dH_{n-1}(y) = \int_{\mathscr{B}_R} |Df| \tag{1.53}$$

Proof. Note from (1.47) applied to f_1 and f_2 respectively, one has for any $g \in C_0^1(C_R;\mathbb{R}^n)$

$$\int_{C_R^+} f_1 \operatorname{div} g\, dx = -\int_{C_R^+} \langle g,\, Df_1\rangle - \int_{\mathscr{B}_R} f^+\, g_n\, dH_{n-1}$$
$$\int_{C_R^-} f_2 \operatorname{div} g\, dx = -\int_{C_R^-} \langle g,\, Df_2\rangle + \int_{\mathscr{B}_R} f^-\, g_n\, dH_{n-1}$$

Notice on C_R^-, while deriving (1.47) for smooth f, one take unit normal $\nu = (0, \cdots, 0, 1)$ pointing upwards to $\mathbb{R}^{n-1}$. Hence the last term involving g_n has opposite signs. One take sum of the above to obtain

$$\int_{C_R} f \operatorname{div} g\, dx = -\int_{C_R^+} \langle g,\, Df_1\rangle - \int_{C_R^-} \langle g,\, Df_2\rangle - \int_{\mathscr{B}_R} (f^+ - f^-)\, g_n\, dH_{n-1} \tag{1.54}$$

Now if require $|g| \le 1$, one has

$$|\int_{C_R} f \operatorname{div} g\, dx| \le \int_{C_R^+} |Df_1| + \int_{C_R^-} |Df_2| + \int_{\mathscr{B}_R} |f^+|\, dH_{n-1} + \int_{\mathscr{B}_R} |f^-|\, dH_{n-1} < \infty$$

Hence $f \in BV(C_R)$. But on the other hand, by definition of distributional gradient Df

$$\int_{C_R} f \operatorname{div} g\, dx = -\int_{C_R} \langle g,\, Df\rangle = -\int_{C_R^+} \langle g,\, Df\rangle - \int_{C_R^-} \langle g,\, Df\rangle - \int_{\mathscr{B}_R} \langle g,\, Df\rangle$$

Notice f coincides with f_1 and f_2 respectively on C_R^+ and C_R^-, hence

$$\int_{C_R} f \operatorname{div} g\, dx = -\int_{C_R^+} \langle g,\, Df_1\rangle - \int_{C_R^-} \langle g,\, Df_2\rangle - \int_{\mathscr{B}_R} \langle g,\, Df\rangle \tag{1.55}$$

Now combining (1.54) and (1.55) gives

$$\int_{\mathscr{B}_R} (f^+ - f^-)\, g_n\, dH_{n-1} = \int_{\mathscr{B}_R} \langle g,\, Df\rangle$$

so

$$\int_{\mathscr{B}_R} |Df| = \sup_{\substack{g \in C_0^1(C_R;\mathbb{R}^n)\\ |g|\le 1}} |\int_{\mathscr{B}_R} \langle g,\, Df\rangle| = \sup_{\substack{g \in C_0^1(C_R;\mathbb{R}^n)\\ |g|\le 1}} |\int_{\mathscr{B}_R} (f^+ - f^-)\, g_n\, dH_{n-1}| = \int_{\mathscr{B}_R} |f^+ - f^-|\, dH_{n-1}$$

where the last equality holds by Riesz Representation. Hence we're done with (1.53). □

1.2.3 Trace on Lipschitz Domains

One has systematic tools to reduce a Domain to C_R. Let $\Omega \subset \mathbb{R}^n$ open with $\partial\Omega$ Lipschitz.

- Since $\partial\Omega$ Lipschitz, for any $x_0 \in \partial\Omega$, there exists a neighborhood around x_0 s.t. the intersection of $\partial\Omega$ and the neighborhood is locally the graph of a Lipschitz function. Due to topology in $\mathbb{R}^n$, one is in fact free to choose the neighborhood as simple geometric objects. Via translation, one may first put $x_0 = 0$ as the origin, then rotate $\partial\Omega$ so that one may choose a cylinder $C(R) = \mathscr{B}_R \times (-\frac{R}{2}, \frac{R}{2})$ with $\mathscr{B}_R$ radius $R > 0$ and height $\frac{R}{2}$, as well as a local Lipschitz function $w : \mathscr{B}_R \subset \mathbb{R}^{n-1} \to (-\frac{R}{2}, \frac{R}{2})$ where the local boundary and interior writes

$$\partial\Omega \cap C(R) = \{(y,\, t) \in C(R) = \mathscr{B}_R \times (-\frac{R}{2},\, \frac{R}{2}) \mid t = w(y)\} \tag{1.56}$$
$$\Omega \cap C(R) = \{(y,\, t) \in C(R) \mid t > w(y)\} \tag{1.57}$$

- One may further flatten out the local boundary by introducing the variables

$$(y,\, \tau) = (y,\, t - w(y)) \in C_R^+ = \mathscr{B}_R \times (0,\, R)$$

  hence for $f \in BV(\Omega \cap C(R))$, one may further define for $g \in BV(C_R^+)$ via

$$g(y,\, \tau) := f(y,\, w(y) + \tau) = f(y,\, t) \tag{1.58}$$

- Apply Theorem 1.2.1 to $g \in BV(C_R^+)$, there exists trace $g^+ \in L^1(\mathscr{B}_R)$. One define $f^+ \in L^1(\partial\Omega \cap C(R))$ for $f \in BV(\Omega \cap C(R))$ as the trace on local Lipschitz boundary via

$$f^+(y,\, w(y)) := g^+(y) \tag{1.59}$$

Theorem 1.2.2 (Construction of Trace). *$\Omega \subset \mathbb{R}^n$ open and bounded with $\partial\Omega$ Lipschitz. $f \in BV(\Omega)$. Then there exists trace $\varphi \in L^1(\partial\Omega)$ s.t. for H_{n-1}-a.e. $x \in \partial\Omega$*

$$\lim_{\rho\to 0} \frac{1}{\rho^n} \int_{B_\rho(x)\cap\Omega} |f(z) - \varphi(x)|\, dz = 0 \tag{1.60}$$

And for any $g \in C_0^1(\mathbb{R}^n;\mathbb{R}^n)$ one has, denoting ν outer unit normal w.r.t. $\partial\Omega$

$$\int_\Omega f \operatorname{div} g\, dx = -\int_\Omega \langle g,\, Df\rangle + \int_{\partial\Omega} \varphi \langle g,\, \nu\rangle\, dH_{n-1} \tag{1.61}$$

Proof. For $\Omega \subset \mathbb{R}^n$ bounded, $\partial\Omega$ is compact. Hence consider open cover $\{C_x(R)\}_{x\in\partial\Omega}$ where $C_x(R)$ is the cylinder s.t. upon translation and rotation, (1.56) and (1.57) holds for x positioned at the origin. There exists finite subcover $\{C_{x_i}(R_i)\}_{i=1}^N$. Given $f \in BV(\Omega)$, upon defining local trace $f_i^+ \in L^1(\partial\Omega \cap C_{x_i}(R_i))$ for each $f|_{C_{x_i}(R_i)}$ as in (1.59), one observe that on their overlaps they must agree H_{n-1}-a.e. due to uniqueness of L^1 limit. Hence $\varphi(x) := f_i^+(x)$ for i s.t. $x \in C_{x_i}(R_i)$ is a well-defined $L^1(\partial\Omega)$ function. Note for any $x \in \partial\Omega$, and for i s.t. $x \in C_{x_i}(R_i)$, there exists $\rho < \frac{R_i}{2}$ s.t. $B_\rho(x) \subset C_{x_i}(R_i)$. Hence (1.60) follows directly from (1.46) as a local behavior. To derive (1.61), one needs partition of unity. Denote $\Gamma_i := C_{x_i}(R_i)$ for $i \geq 1$ and $\Gamma_0 \subset\subset \Omega$ chosen s.t. $\overline{\Omega} \subset \bigcup_{i=0}^N \Gamma_i$ is open cover. One may choose a smooth partition of unity subordinate to $\{\Gamma_i\}_0^N$ s.t.

$$0 \leq \phi_i \leq 1, \qquad \operatorname{supp}\phi_i \subset \Gamma_i, \qquad \sum_{i=0}^N \phi_i = 1 \text{ in } \overline{\Omega}$$

Hence $f = \sum_{i=0}^N f\phi_i$ in Ω and $\varphi = \sum_{i=1}^N \varphi\phi_i$ on $\partial\Omega$ since $\Gamma_0 \subset\subset \Omega$. By definition of distributional derivative $D(f\phi_0) \in D'$ and that $\operatorname{supp} f\phi_0 \subset \Gamma_0 \subset\subset \Omega$, for any $g \in C_0^1(\mathbb{R}^n;\mathbb{R}^n)$

$$\int_\Omega f\,\phi_0 \operatorname{div} g\, dx = \int f\,\phi_0 \operatorname{div} g\, dx = -\int \langle g,\, D(f\phi_0)\rangle = -\int_\Omega \langle g,\, D(f\phi_0)\rangle \tag{1.62}$$

while for $i = 1, \cdots, N$, one apply flattening boundary and then (1.47) on each $C_{R_i}^+$ to obtain

$$\int_\Omega f\phi_i \operatorname{div} g\, dx = -\int_\Omega \langle g,\, D(f\phi_i)\rangle + \int_{\partial\Omega} \varphi\phi_i \langle g,\, \nu\rangle\, dH_{n-1} \tag{1.63}$$

Hence summing up (1.62) and (1.63) gives (1.61). □

Proposition 1.2.3 (Approximation in BV implies Approximation in Trace). *$\Omega \subset \mathbb{R}^n$ open and bounded, $\partial\Omega$ Lipschitz. $f \in BV(\Omega)$. If $\{f_j\} \subset BV(\Omega)$ s.t. $f_j \to f$ in $L^1(\Omega)$ and*

$$\lim_{j\to\infty} \int_\Omega |Df_j| = \int_\Omega |Df| \tag{1.64}$$

then, letting φ_j be trace for f_j and φ trace for f

$$\lim_{j\to\infty} \int_{\partial\Omega} |\varphi_j - \varphi|\, dH_{n-1} = 0 \tag{1.65}$$

Remark 1.2.1. *Let $\Omega \subset \mathbb{R}^n$ open and bounded, $\partial\Omega$ Lipschitz. $f \in BV(\Omega)$.*

- *By smooth approximation Theorem 1.1.2, there exists $\{f_j\} \subset C^\infty(\Omega)$ s.t. $\|f_j - f\|_{L^1(\Omega)} \to 0$ and $\lim_{j\to 0} \int_\Omega |Df_j|\, dx = \int_\Omega |Df|$. As in Proposition 1.2.1, or essentially (1.18), letting φ_j be trace for f_j and φ trace for f, one has $\varphi_j = \varphi$ for any j.*
- *Let $A \subset\subset \Omega$ open with ∂A Lipschitz. Then $f|_A \in BV(A)$ and $f|_{\Omega\setminus\overline{A}}$, hence denote $f_A^-, f_A^+ \in L^1(\partial A)$ as their trace respectively.*
  1. *One has immediately via differentiation (1.60) that for H_{n-1}-a.e. $x \in \partial A$*

$$\lim_{\rho\to 0} \frac{1}{\rho^n} \int_{B_\rho(x)\cap A} |f(z) - f_A^-(x)|\, dz = 0 \qquad \lim_{\rho\to 0} \frac{1}{\rho^n} \int_{B_\rho(x)\cap(\Omega\setminus\overline{A})} |f(z) - f_A^+(x)|\, dz = 0 \tag{1.66}$$

  2. *Via Extension property for BV Proposition 1.2.2, denoting ν as outer unit normal w.r.t. ∂A, one has important characterisation for the measures $|Df|$ and Df on ∂A*

$$\int_{\partial A} |Df| = \int_{\partial A} |f_A^+ - f_A^-|\, dH_{n-1}(y) \tag{1.67}$$

$$\int_{\partial A} Df = \int_{\partial A} \left(f_A^+ - f_A^-\right) \nu\, dH_{n-1}(y) \tag{1.68}$$

In particular, let $\Omega = B_R$ and $A = B_\rho$ for $\rho < R$, and denote f_ρ^-, $f_\rho^+ \in L^1(\partial B_\rho)$ as trace for $f|_{B_\rho}$ and $f|_{B_R \setminus \overline{B}_\rho}$ respectively. One has, for some N_1, $N_2 \subset \mathbb{R}$ set measure 0

$$\lim_{\substack{t\to\rho^- \\ t\notin N_1}} \int_{\partial B_1} |f(tx) - f_\rho^-(\rho x)|\, dH_{n-1}(x) = 0 \qquad \lim_{\substack{t\to\rho^+ \\ t\notin N_2}} \int_{\partial B_1} |f(tx) - f_\rho^+(\rho x)|\, dH_{n-1}(x) = 0 \tag{1.69}$$

Proof. It suffices to prove for f_ρ^-. Notice, by a change of variables, for any $\frac{\rho}{2} < t < \rho$

$$\begin{aligned}\int_{\partial B_1} |f(tx) - f_\rho^-(\rho x)|\, dH_{n-1}(x) &= \frac{1}{\rho^n} \int_{\partial B_\rho} |f(\frac{t}{\rho}x) - f_\rho^-(x)|\, dH_{n-1}(x) \\ &\leq \frac{1}{\rho^n} \frac{1}{(\rho-t)^n} \int_{\partial B_\rho} \int_{B_{2(\rho-t)}(x)\cap B_\rho} |f(z) - f_\rho^-(x)|\, dz\, H_{n-1}(x)\end{aligned}$$

where the last inequality holds for a.e. t. Denote the set that it fails by N_1. Now since $f \in L^1(B_R)$, one may apply DCT and use the inner part of (1.66)

$$\begin{aligned}\limsup_{\substack{t\to\rho^- \\ t\notin N_1}} \int_{\partial B_1} |f(tx) - f_\rho^-(\rho x)|\, dH_{n-1}(x) &\leq \limsup_{\substack{t\to\rho^- \\ t\notin N_1}} \frac{1}{\rho^n} \int_{\partial B_\rho} \frac{1}{(\rho-t)^n} \int_{B_{2(\rho-t)}(x)\cap B_\rho} |f(z) - f_\rho^-(x)|\, dz\, H_{n-1}(x) \\ &\leq \frac{1}{\rho^n} \int_{\partial B_\rho} \left(\lim_{\substack{t\to\rho^- \\ t\notin N_1}} \frac{1}{(\rho-t)^n} \int_{B_{2(\rho-t)}(x)\cap B_\rho} |f(z) - f_\rho^-(x)|\, dz \right) H_{n-1}(x) \\ &= 0\end{aligned}$$

□

Also, since $f \in BV(\Omega)$, $|Df|$ is of finite measure. Due to countable additivity of measure for $|Df|$, for a.e. ρ, one has $\int_{\partial B_\rho} |Df| = 0$, hence

$$f_\rho^+(x) = f(x) = f_\rho^-(x) \qquad \text{for } H_{n-1} - \text{a.e. } x \in \partial B_\rho \text{ for a.e. } \rho \tag{1.70}$$

- *Let $A \subset \Omega$ open with ∂A Lipschitz, and $f \in BV(A)$. One may extend f to Ω by $F := \begin{cases} f & \text{in } A \\ 0 & \text{in } \Omega \setminus A \end{cases}$ hence denoting F_A^-, $F_A^+ \in L^1(\partial A)$ as trace for $F|_A$, $F|_{\Omega\setminus\overline{A}}$, one has $F_A^- = f_A^-$ as trace of f on ∂A, and $F_A^+ = 0$.*

  1. *from (1.67)*

  $$\int_\Omega |DF| - \int_A |Df| = \int_{\Omega\cap\partial A} |DF| = \int_{\Omega\cap\partial A} |f_A^-|\, dH_{n-1} \tag{1.71}$$

  2. *from (1.68), denoting ν as inner unit normal w.r.t. ∂A*

  $$\int_\Omega DF - \int_A Df = \int_{\Omega\cap\partial A} DF = \int_{\Omega\cap\partial A} f_A^-\, \nu\, dH_{n-1} \tag{1.72}$$

In particular, one may further compute 3 perimeters for subsets of Caccioppoli set w.r.t. some ball. Let $\Omega = B_R$ and $A = B_\rho$ for $\rho < R$, and $f = \varphi_E$ for $E \subset \mathbb{R}^n$ Caccioppoli. Then $F = \varphi_{E\cap B_\rho}$. Due to (1.70), for a.e. ρ, $\varphi_E = \varphi_{E,\rho}^-$ for H_{n-1}-a.e. $x \in \partial B_\rho$. Note $\partial B_\rho \cap B_R = \partial B_\rho$, so

1. *from (1.71)*

$$P(E\cap B_\rho, B_R) = P(E, B_\rho) + H_{n-1}(E\cap\partial B_\rho) \qquad \text{for a.e. } \rho \text{ s.t. (1.70) holds} \tag{1.73}$$

2. *similarily, from (1.72), denoting ν as inner unit normal w.r.t. ∂B_ρ*

$$\int_{B_R} D\varphi_{E\cap B_\rho} = \int_{B_\rho} D\varphi_E + \int_{\partial B_\rho} \varphi_E\, \nu\, dH_{n-1} \qquad \text{for a.e. } \rho \text{ s.t. (1.70) holds} \tag{1.74}$$

Now let $A = B_R \setminus \overline{B}_\rho$, then $F = \varphi_{E\cap(B_R\setminus\overline{B}_\rho)}$, so for a.e. ρ, $\varphi_E = \varphi_{E,\rho}^+$ for H_{n-1}-a.e. $x \in \partial B_\rho$

$$P(E\setminus\overline{B}_\rho, B_R) = P(E, B_R\setminus\overline{B}_\rho) + H_{n-1}(E\cap\partial B_\rho) \qquad \text{for a.e. } \rho \text{ s.t. (1.70) holds} \tag{1.75}$$

Furthermore for A as above, $B_R\setminus(E\cap(B_R\setminus\overline{B}_\rho))=(B_R\setminus E)\cap(B_R\setminus\overline{B}_\rho)$, then using that mutual disjoint sets share same perimeter

$$P((B_R\setminus E)\cap(B_R\setminus\overline{B}_\rho),\,B_R)=P(E\cap(B_R\setminus\overline{B}_\rho),\,B_R)=P(E\setminus\overline{B}_\rho,\,B_R)$$

one has, again by mutual disjoint sets sharing same perimeter

$$\begin{aligned}P(E\cup\overline{B}_\rho,\,B_R)&=P(B_R\setminus(E\cup\overline{B}_\rho),\,B_R)=P((B_R\setminus E)\cap(B_R\setminus\overline{B}_\rho),\,B_R)=P(E\setminus\overline{B}_\rho,\,B_R)\\&=P(E,\,B_R\setminus\overline{B}_\rho)+H_{n-1}(E\cap\partial B_\rho)\qquad \textit{for a.e. } \rho \textit{ s.t. (1.70) holds}\end{aligned}\tag{1.76}$$

Hence one may measure perimeter of subsets for E in big ball using perimeter of E in small balls and the boundary quantity $H_{n-1}(E\cap\partial B_\rho)$ via (1.73), (1.75) and (1.76).

1.2.4 Converse to Trace Construction

Theorem 1.2.3 (Converse to Trace Construction)**.** *Let $\varphi\in L^1(\mathscr{B}_R)$ for $R>0$ and compactly supported. For any $\varepsilon>0$, there exists $f\in W^{1,1}(C_R^+)$ s.t. φ is trace of f and*

$$\int_{C_R^+}|f|\,dx\le\varepsilon\int_{\mathscr{B}_R}|\varphi|\,dH_{n-1}\tag{1.77}$$

$$\int_{C_R^+}|Df|\,dx\le(1+\varepsilon)\int_{\mathscr{B}_R}|\varphi|\,dH_{n-1}\tag{1.78}$$

Proof. There exists $\{\varphi_j\}\subset C^\infty(\mathscr{B}_R)$ s.t. $\|\varphi_j-\varphi\|_{L^1(\mathscr{B}_R)}\to 0$ with $\varphi_0=0$, $\|\varphi_j\|_{L^1(\mathscr{B}_R)}\le 2\|\varphi\|_{L^1(\mathscr{B}_R)}$ and

$$\int_{\mathscr{B}_R}|\varphi_j-\varphi_{j+1}|\,dH_{n-1}\le 2^{-j-1}\left(1+\frac{\varepsilon}{2}\right)\int_{\mathscr{B}_R}|\varphi|\,dH_{n-1}\implies\sum_{j=0}^\infty\|\varphi_j-\varphi\|_{L^1(\mathscr{B}_R)}\le\left(1+\frac{\varepsilon}{2}\right)\|\varphi\|_{L^1(\mathscr{B}_R)}$$

Now one may construct f with support on neighborhood of $\mathscr{B}_R$. Let $\{t_k\}\subset(0,R)$ be strictly decreasing sequence to 0. Define $f:C_R^+\to\mathbb{R}$ s.t. for $x\in\mathscr{B}_R$, $t\in(0,R)$

$$f(x,\,t):=\begin{cases}0 & \textit{if } t>t_0\\ \frac{t-t_{k+1}}{t_k-t_{k+1}}\varphi_k(x)+\frac{t_k-t}{t_k-t_{k+1}}\varphi_{k+1}(x) & \textit{if } t_k\ge t>t_{k+1} \textit{ for } k\ge 0\end{cases}$$

Hence one may calculate for any $t_k\ge t>t_{k+1}$ for $k\ge 0$

$$\begin{aligned}|D_if|&\le|D_i\varphi_k(x)|+|D_i\varphi_{k+1}(x)|\quad 1\le i\le n-1\\|D_nf|&\le\frac{1}{t_k-t_{k+1}}|\varphi_k(x)-\varphi_{k+1}(x)|\end{aligned}$$

Hence one calculate $\int_{C_R^+}|f|\,dx$ and $\int_{C_R^+}|Df|\,dx$ s.t.

$$\begin{aligned}\int_{C_R^+}|f|\,dx&=\int_0^R\int_{\mathscr{B}_R}|f|\,dH_{n-1}(x)\,dt=\sum_{k=0}^\infty\int_{t_{k+1}}^{t_k}\int_{\mathscr{B}_R}|f|\,dH_{n-1}(x)\,dt\\&\le\sum_{k=0}^\infty\int_{t_{k+1}}^{t_k}\left(\|\varphi_k\|_{L^1(\mathscr{B}_R)}+\|\varphi_{k+1}\|_{L^1(\mathscr{B}_R)}\right)dt=\sum_{k=0}^\infty\left(\|\varphi_k\|_{L^1(\mathscr{B}_R)}+\|\varphi_{k+1}\|_{L^1(\mathscr{B}_R)}\right)(t_k-t_{k+1})\\&\le 4\|\varphi\|_{L^1(\mathscr{B}_R)}\sum_{k=0}^\infty(t_k-t_{k+1})=4t_0\|\varphi\|_{L^1(\mathscr{B}_R)}\\\int_{C_R^+}|Df|\,dx&=\sum_{k=0}^\infty\int_{t_{k+1}}^{t_k}\int_{\mathscr{B}_R}|Df|\,dH_{n-1}(x)\,dt\le\sum_{k=0}^\infty\int_{t_{k+1}}^{t_k}\sum_{i=1}^n\int_{\mathscr{B}_R}|D_if|\,dH_{n-1}(x)\,dt\\&\le\sum_{k=0}^\infty\int_{t_{k+1}}^{t_k}\left(\sum_{i=1}^{n-1}\left(\|D_i\varphi_k\|_{L^1(\mathscr{B}_R)}+\|D_i\varphi_{k+1}\|_{L^1(\mathscr{B}_R)}\right)+\frac{1}{t_k-t_{k+1}}\|\varphi_k-\varphi_{k+1}\|_{L^1(\mathscr{B}_R)}\right)dt\\&\le\sum_{k=0}^\infty\left(\left(\|D\varphi_k\|_{L^1(\mathscr{B}_R)}+\|D\varphi_{k+1}\|_{L^1(\mathscr{B}_R)}\right)(t_k-t_{k+1})+\|\varphi_k-\varphi_{k+1}\|_{L^1(\mathscr{B}_R)}\right)\\&\le\sum_{k=0}^\infty\left(\|D\varphi_k\|_{L^1(\mathscr{B}_R)}+\|D\varphi_{k+1}\|_{L^1(\mathscr{B}_R)}\right)(t_k-t_{k+1})+\left(1+\frac{\varepsilon}{2}\right)\|\varphi\|_{L^1(\mathscr{B}_R)}\end{aligned}$$

But one is left to choose t_k freely. Hence choose t_k s.t. $4t_0 < \varepsilon$ and for $k \geq 0$

$$(t_k - t_{k+1}) \leq \frac{\varepsilon \|\varphi\|_{L^1(\mathscr{B}_R)}}{1 + \|D\varphi_k\|_{L^1(\mathscr{B}_R)} + \|D\varphi_{k+1}\|_{L^1(\mathscr{B}_R)}} 2^{-k-2}$$

Hence one obtain (1.77) and (1.78), whence $f \in W^{1,1}(C_R^+)$. To see φ really is trace for f, denote $f_t(x) := f(x,t)$ and compute for $t_k \geq t > t_{k+1}$, following construction in Theorem 1.2.1 and DCT

$$\int_{\mathscr{B}_R} |f_t(x) - \varphi(x)| \, dH_{n-1}(x) \leq \int_{\mathscr{B}_R} |\frac{t - t_{k+1}}{t_k - t_{k+1}} \varphi_k(x) - \varphi(x)| \, dH_{n-1}(x) + \int_{\mathscr{B}_R} |\frac{t_k - t}{t_k - t_{k+1}} \varphi_{k+1}(x) - \varphi(x)| \, dH_{n-1}(x) \overset{k \to \infty}{\to} 0$$

Hence by uniqueness of L^1 limits, φ is indeed trace for f.

□

Theorem 1.2.4 (Converse to Trace Construction)**.** *$\Omega \subset \mathbb{R}^n$ open bounded, $\partial\Omega$ Lipschitz. $\varphi \in L^1(\partial\Omega)$. Then for any $\varepsilon > 0$, there exists $f \in W^{1,1}(\Omega)$ s.t. φ is trace of f and*

$$\int_\Omega |f| \, dx \leq \varepsilon \int_{\partial\Omega} |\varphi| \, dH_{n-1} \tag{1.79}$$

$$\int_\Omega |Df| \, dx \leq A \int_{\partial\Omega} |\varphi| \, dH_{n-1} \tag{1.80}$$

for $A = A(\partial\Omega)$ but independent of f, φ, ε. If moreover $\partial\Omega$ is C^1, one may choose $A = (1 + \varepsilon)$. Also, f may be taken to be supported on arbitrary small neighborhood of $\partial\Omega$ by controlling t_0 via ε.

Chapter 2

Reduced Boundary

2.1 Construction and Properties

As a preliminary, one finds substitution for general Borel sets so that their measure theoretic boundary and topological boundary agree. We work with sets satisfying Lemma 2.1.1 from later on.

Lemma 2.1.1. *Let $E \subset \mathbb{R}^n$ Borel. Then there exists $\tilde{E}$ Borel s.t. $|\tilde{E}\Delta E| = 0$ differ by Lebesgue measure 0 and*

$$0 < |\tilde{E} \cap B_\rho(x)| < \omega_n \rho^n \quad \textit{for any } \rho > 0 \textit{ and } x \in \partial\tilde{E} \tag{2.1}$$

Proof. Define

$$E_0 := \{x \in \mathbb{R}^n \mid \textit{ there exists } \rho > 0 \textit{ s.t. } |E \cap B_\rho(x)| = 0\}$$
$$E_1 := \{x \in \mathbb{R}^n \mid \textit{ there exists } \rho > 0 \textit{ s.t. } |E \cap B_\rho(x)| = |B_\rho(x)| = \omega_n\rho^n\}$$

One see both E_0 and E_1 are open. For $x \in E_0$, take $\rho > 0$ s.t. $|E \cap B_\rho(x)| = 0$. Then for any $y \in B_\rho(x)$, let $\rho_0 := \rho - |x-y|$, so $B_{\rho_0}(y) \subset B_\rho(x)$ hence $|E \cap B_{\rho_0}(y)| = 0$. Due to existence of ρ_0, $y \in E_0$, i.e., the neighborhood $B_\rho(x) \subset E_0$. So E_0 open. For $x \in E_1$, there exists $\rho > 0$ s.t. $|E \cap B_\rho(x)| = |B_\rho(x)|$, i.e., $|B_\rho(x) \cap E^c| = 0$. Again, for any $y \in B_\rho(x)$, let $\rho_0 := \rho - |x-y|$, so $B_{\rho_0}(y) \subset B_\rho(x)$, thus $|B_{\rho_0}(y) \cap E^c| = 0$. Hence $y \in E_1$, we have $B_\rho(x) \subset E_1$, so E_1 is open. One may further show that $|E_0 \cap E| = 0$. Since for any $x \in E_0$, one may choose ρ_x s.t. $|E \cap B_{\rho_x}(x)| = 0$, and it indeed covers $E_0 \subset \bigcup_{x\in E_0} B_{\rho_x}(x)$, we may choose sequence $\{x_j\} \subset E_0$ as index for covering. One compute, due to $|B_{\rho_{x_j}}(x_j) \cap E| = 0$ for any j

$$|E_0 \cap E| \leq |\bigcup_{j=1}^{\infty} B_{\rho_{x_j}}(x_j) \cap E| \leq \sum_{j=1}^{\infty} |B_{\rho_{x_j}}(x_j) \cap E| = 0$$

Similarly, $|E_1 \setminus E| = 0$ by replacing E in above computation with E^c. Since E_0, E_1 open, $\tilde{E} := (E \cup E_1) \setminus E_0$ is Borel. And indeed one has $|\tilde{E}\Delta E| = 0$ via the following

$$|E \setminus \tilde{E}| = |E \cap ((E\cup E_1)\setminus E_0)^c| = |E \cap ((E \cup E_1)^c \cup E_0)\,| = |(E \cap E^c \cap E_1^c) \cup (E \cap E_0)| = |E_0 \cap E| = 0$$
$$|\tilde{E} \setminus E| = |(E \cup E_1) \cap E_0^c \cap E^c| = |(E \cap E_0^c \cap E^c) \cup (E_1 \cap E_0^c \cap E^c)| \leq |E_1 \setminus E| = 0$$

Now for any $x \in \partial\tilde{E}$, since E_0, E_1 open, $x \notin E_0 \cup E_1$. Hence for any $\rho > 0$, (2.1) holds. □

2.1.1 Reduced Boundary and Uniform Density Estimate

Definition 2.1.1 (Reduced Boundary)**.** *Given $E \subset \mathbb{R}^n$ Caccioppoli. $x \in \partial^* E$ reduced boundary if*

$$\int_{B_\rho(x)} |D\varphi_E| > 0 \quad \textit{for any } \rho > 0 \tag{2.2}$$

and hence, defining

$$\nu_\rho(x) := \frac{\int_{B_\rho(x)} D\varphi_E}{\int_{B_\rho(x)} |D\varphi_E|} \quad \textit{for any } \rho > 0 \tag{2.3}$$

One require the limits $\lim_{\rho\to 0} \nu(x)$ exists and has length 1

$$\nu(x) := \lim_{\rho\to 0} \nu(x) = \lim_{\rho \to 0} \frac{\int_{B_\rho(x)} D\varphi_E}{\int_{B_\rho(x)} |D\varphi_E|} \tag{2.4}$$

$$|\nu(x)| = 1 \tag{2.5}$$

i.e., $\partial^* E := \{x \in \partial E \mid$ (2.2) holds for any $\rho > 0$, and the limiting object (2.3) satisfies (2.4) and (2.5)$\}$

Recall the Lebesgue-Besicovitch differentiation.

Lemma 2.1.2 (Lebesgue-Besicovitch differentiation)**.** *μ_1, μ_2 Borel measures on $\mathbb{R}^n$, then*

$$D_{\mu_2}\mu_1 := \lim_{\rho\to 0}\frac{\mu_1(B_\rho(x))}{\mu_2(B_\rho(x))}$$

is defined μ_2-a.e. on $\mathbb{R}^n$, and $D_{\mu_2}\mu_1 \in L^1_{loc}(\mathbb{R}^n, \mu_2)$. If furthermore, $\mu_1 \ll \mu_2$, i.e., μ_1 is absolutely continuous w.r.t. μ_2 in the sense that $\mu_2(E) = 0$ implies $\mu_1(E)$ for any $E \subset \mathbb{R}^n$ Borel, then we write

$$\mu_1 = D_{\mu_2}\mu_1 \cdot \mu_2 \qquad \textit{on all Borel sets}$$

Remark 2.1.1. *Note $D\varphi_E$ is indeed absolutely continuous w.r.t. $|D\varphi_E|$. Hence apply Lemma 2.1.2, one has*

$$\nu(x) := \lim_{\rho\to 0}\frac{\int_{B_\rho(x)} D\varphi_E}{\int_{B_\rho(x)}|D\varphi_E|} \textit{ exists and } |\nu(x)| = 1 \qquad |D\varphi_E| - a.e.\ x \in \mathbb{R}^n \tag{2.6}$$

and the following measures agree

$$D\varphi_E = \nu|D\varphi_E| \qquad \textit{on all Borel sets} \tag{2.7}$$

Example 2.1.1. *One has 2 examples. One for smooth boundary and one for Lipschitz.*

- *Let $E \subset \mathbb{R}^n$ be bounded, Caccioppoli with C^2 boundary ∂E. Then $\partial^* E = \partial E$.*

  Proof. Let $A = E$ and $f = \varphi_E$ in (1.68), one has via Extension property for $\varphi_E \in BV(\mathbb{R}^n)$ that

  $$D\varphi_E = \nu dH_{n-1} \quad \textit{on } \partial E$$

  where ν denote inner unit normal w.r.t. ∂E. And because supp$D\varphi_E \subset \partial E$, one writes for any $\rho > 0$

  $$\int_{B_\rho(x)} D\varphi_E = \int_{B_\rho(x)\cap\partial E} \nu\, dH_{n-1}$$

  while C^2 boundary ensure via (1.4) that

  $$\int_{B_\rho(x)} |D\varphi_E| = H_{n-1}(B_\rho(x)\cap\partial E)$$

  hence one has explicit formula for ν_ρ

  $$\nu_\rho(x) = \frac{\int_{B_\rho(x)\cap\partial E}\nu\, dH_{n-1}}{H_{n-1}(B_\rho(x)\cap\partial E)} \qquad \textit{for any } x \in \partial E$$

  Since $\nu \in C^1(\partial E;\mathbb{R}^n)$, differentiation gives $\lim_{\rho\to 0}\nu_\rho(x) = \nu(x)$ for any $x \in \partial E$. Hence $|\nu| = 1$ as inherited. □

- *Let $E = (0,1)\times(0,1) \subset \mathbb{R}^2$. Notice except for the four corners, the boundaries are piecewise C^∞, hence these parts belong to $\partial^* E$. Now for any corner x, one may compute*

  $$|\nu(x)| = \lim_{\rho\to 0}\frac{|\int_{B_\rho(x)} D\varphi_E|}{\int_{B_\rho(x)}|D\varphi_E|} = \frac{1}{\sqrt{2}}$$

  Hence the four corners do not belong to $\partial^* E$.

One has Uniform Density estimates, which says bounded oscillation in normal directions at a given boundary point $x \in \partial E$ prevents densities of E and E^c from disappearing under blow-up limit. In particular, if $x \in \partial^* E$, it indeed satisfies our assumption, so uniform density estimate holds. For simplicity, let $0 \in \partial E$ via translation.

Theorem 2.1.1 (Uniform Density Estimates)**.** *$E \subset \mathbb{R}^n$ be Caccioppoli and $0 \in \partial E$. If there exists $\rho_0 > 0$ and $q > 0$ constants s.t. for any $\rho < \rho_0$*

$$\int_{B_\rho} |D\varphi_E| > 0$$

$$|\nu_\rho(0)| = \left|\frac{\int_{B_\rho} D\varphi_E}{\int_{B_\rho}|D\varphi_E|}\right| \geq q > 0 \tag{2.8}$$

Then for any $\rho < \rho_0$, one has uniform estimates on the density

$$\frac{|E \cap B_\rho|}{\rho^n} \geq C_1(n, q) > 0 \tag{2.9}$$

$$\frac{|E^c \cap B_\rho|}{\rho^n} \geq C_2(n, q) > 0 \tag{2.10}$$

$$0 < C_3(n, q) \leq \frac{\int_{B_\rho} |D\varphi_E|}{\rho^{n-1}} \leq C_4(n, q) < \infty \tag{2.11}$$

for constants C_1, C_2, C_3, C_4 only relevant to n, q.

Proof. Since E Caccioppoli, $\varphi_E \in BV(B_{\rho_0})$. Denoting ν as inner unit normal w.r.t. ∂B_ρ one has via (1.74)

$$\int D\varphi_{E\cap B_\rho} = \int_{B_\rho} D\varphi_E + \int_{\partial B_\rho} \varphi_E\, \nu \, dH_{n-1} \qquad \text{for a.e. } \rho < \rho_0$$

evaluate the vector-valued measure on some constant unit vector $e \in \mathbb{S}^{n-1}$ gives, for ρ s.t. (1.74) holds

$$0 = -\int \operatorname{div}(e)\, \varphi_{E\cap B_\rho} = \int \langle e, D\varphi_{E\cap B_\rho}\rangle = \int_{B_\rho} \langle e, D\varphi_E\rangle + \int_{\partial B_\rho} \varphi_E\, \nu \cdot e dH_{n-1}$$

Hence for any $e \in \mathbb{S}^{n-1}$

$$\left|\int_{B_\rho} \langle e, D\varphi_E\rangle\right| = \left|\int_{\partial B_\rho} \varphi_E\, \nu \cdot e dH_{n-1}\right| \leq \int_{\partial B_\rho} \varphi_E \, dH_{n-1} = H_{n-1}(E \cap \partial B_\rho) \leq C\rho^{n-1}$$

taking supremum on LHS and using Riesz Representation yields

$$\left|\int_{B_\rho} D\varphi_E\right| \leq H_{n-1}(E \cap \partial B_\rho) \tag{2.12}$$

Using (2.12) and (2.8) further gives

$$\int_{B_\rho} |D\varphi_E| \leq \frac{1}{q}\left|\int_{B_\rho} D\varphi_E\right| \leq C_4 \rho^{n-1} \quad \text{for a.e. } \rho < \rho_0 \text{ s.t. (1.74) holds}$$

Now using continuity from above of the measure $|D\varphi_E|$, we conclude the second part to (2.11) for all $\rho < \rho_0$. Now, using (1.73) and similar reasons as above, one has

$$\begin{aligned} P(E \cap B_\rho) &= P(E, B_\rho) + H_{n-1}(E \cap \partial B_\rho) \quad \text{for a.e. } \rho < \rho_0 \\ &= \int_{B_\rho} |D\varphi_E| + \int_{\partial B_\rho} \varphi_E \, dH_{n-1} \leq \left(\frac{1}{q} + 1\right) \int_{\partial B_\rho} \varphi_E \, dH_{n-1} \end{aligned}$$

Since $E \cap B_\rho$ is bounded Caccioppoli, via isoperimetric inequality (1.40) and noting $P(E \cap B_\rho) = \int |D\varphi_{E\cap B_\rho}|$

$$|E \cap B_\rho|^{\frac{n-1}{n}} \leq \left(\frac{1}{q} + 1\right) C(n) \int_{\partial B_\rho} \varphi_E \, dH_{n-1} \tag{2.13}$$

for some $C(n)$ from (1.40). Notice by coarea formula, denoting $g(\rho) = |E \cap B_\rho|$

$$g(R) = |E \cap B_R| = \int_{B_R} \varphi_E \, dx = \int_0^R \int_{\partial B_\rho} \varphi_E \, dH_{n-1} \, d\rho \implies g'(\rho) = \int_{\partial B_\rho} \varphi_E \, dH_{n-1}$$

Hence (2.13) writes

$$g(\rho)^{\frac{n-1}{n}} \leq \left(\frac{1}{q} + 1\right) C(n) g'(\rho) \implies \rho \leq \left(\frac{1}{q} + 1\right) C(n)\, n\, g(\rho)^{\frac{1}{n}} \implies \left(\frac{1}{C(n)\, n\left(\frac{1}{q} + 1\right)}\right)^n \leq \frac{|E \cap B_\rho|}{\rho^n}$$

denoting $C_1 := \left(\frac{1}{C(n)n\left(\frac{1}{q}+1\right)}\right)^n$ and using continuity from below of the measure $|E \cap B_\rho|$ in ρ, one conclude (2.9) for every $\rho < \rho_0$. Note for E^c, $D\varphi_{E^c} = -D\varphi_E$ due to for any $g \in C_0^1(\mathbb{R}^n; \mathbb{R}^n)$

$$\int \langle g, D\varphi_{E^c}\rangle = -\int \varphi_{E^c} \operatorname{div}(g) \, dx = -\int (1 - \varphi_E) \operatorname{div}(g) \, dx = \int \varphi_E \operatorname{div}(g) \, dx = -\int \langle g, D\varphi_E\rangle$$

whence $|D\varphi_E| = |D\varphi_{E^c}|$ and the above same argument runs with $C_2 = C_1$, resulting in (2.10). To see first part to (2.11), notice from (2.9) and (2.10), one has

$$C_1\rho^n \le \min\{|E\cap B_\rho|,\, |E^c\cap B_\rho|\} \implies C_1^{\frac{n-1}{n}}\rho^{n-1} \le \min\{|E\cap B_\rho|,\, |E^c\cap B_\rho|\}^{\frac{n-1}{n}}$$

Hence applying Poincaré inequality (1.41) one has, for some $\tilde{C}(n) > 0$

$$C_1^{\frac{n-1}{n}}\rho^{n-1} \le \tilde{C}(n)\int_{B_\rho}|D\varphi_E| \implies 0 < \frac{C_1^{\frac{n-1}{n}}}{\tilde{C}(n)} \le \frac{1}{\rho^{n-1}}\int_{B_\rho}|D\varphi_E|$$

define $C_3 := \frac{C_1^{\frac{n-1}{n}}}{\tilde{C}(n)}$ yields the first part of (2.11). □

2.1.2 Blow-up Limit

One define the tangent plane and half spaces for given $z \in \partial^* E$ (hence $\nu(z)$ is well-defined and $|\nu(z)| = 1$)

- Tanget Hyperplane to $\partial^* E$ at z is $T(z) := \{x \in \mathbb{R}^n \mid \langle \nu(z),\, x - z\rangle = 0\}$
- Half spaces to $\partial^* E$ at z on the same and opposite side with $\nu(z)$ are respectively

$$T^+(z) := \{x \in \mathbb{R}^n \mid \langle \nu(z),\, x - z\rangle > 0\}$$
$$T^-(z) := \{x \in \mathbb{R}^n \mid \langle \nu(z),\, x - z\rangle < 0\}$$

One may now show that the blowup limit of a point in reduced boundary actually converges to the half space on the same side as the outer normal. For simplicity, via translation and rotation, one assume $0 \in \partial^* E$, and the outer normal $\nu(0)$ is parallel to the x_1-axis that points towards $-\infty$. One wish to obtain the limit $T^+(0)$.

Theorem 2.1.2 (Blow-up Limit of Reduced Boundary)**.** *$E \subset \mathbb{R}^n$ Caccioppoli. $0 \in \partial^* E$ with $\nu(0) = (-1, 0, \cdots, 0)$. For any $t > 0$, define the set for blowup*

$$E_t := \{x \in \mathbb{R}^n \mid t\,x \in E\} \tag{2.14}$$

Then there exists a subsequence $t_j \to 0^+$ s.t. $E_j := E_{t_j} \to T^+ := T^+(0)$ in $L^1_{loc}(\mathbb{R}^n)$ sense. Moreover, for every open set $A \subset \mathbb{R}^n$ s.t. $H_{n-1}(\partial A \cap T(0)) = 0$

$$\lim_{t_j\to 0}\int_A |D\varphi_{E_j}| = \int_A |D\varphi_{T^+}| = H_{n-1}(T(0)\cap A) \tag{2.15}$$

Proof. One wish to extract a convergent subsequence using compactness argument. First note in our setting, the targeting limit is $T^+ = \{x \in \mathbb{R}^n \mid x_1 < 0\}$. Fix $\rho > 0$. Now by change of variables, for any $g \in C_0^1(B_\rho; \mathbb{R}^n)$, write $\tilde{g}(x) := g(x/t)$

$$\begin{aligned}\int_{B_\rho}\langle g,\, D\varphi_{E_t}\rangle &= -\int_{B_\rho}\operatorname{div}(g(x))\,\varphi_{E_t}(x)\,dx = -\int_{B_\rho}\operatorname{div}(\tilde{g}(tx))\,\varphi_E(tx)\,dx\\ &= -\int_{B_\rho} t\operatorname{div}(\tilde{g})(tx)\,\varphi_E(tx)\,dx = -\frac{1}{t^{n-1}}\int_{B_{t\rho}}\operatorname{div}(\tilde{g})(y)\,\varphi_E(y)\,dy\\ &= \frac{1}{t^{n-1}}\int_{B_{t\rho}}\langle\tilde{g},\, D\varphi_E\rangle \implies \int_{B_\rho} D\varphi_{E_t} = \frac{1}{t^{n-1}}\int_{B_{t\rho}} D\varphi_E\end{aligned} \tag{2.16}$$

And by considering total variation, one has

$$\int_{B_\rho}|D\varphi_{E_t}| = \frac{1}{t^{n-1}}\int_{B_{t\rho}}|D\varphi_E| \tag{2.17}$$

With tools (2.16) and (2.17), one proceeds in two directions. First, making use of $0 \in \partial^* E$, in particular (2.4)

$$\lim_{t\to 0}\frac{1}{\int_{B_\rho}|D\varphi_{E_t}|}\begin{pmatrix}\int_{B_\rho} D_1\varphi_{E_t}\\ \int_{B_\rho} D_2\varphi_{E_t}\\ \vdots\\ \int_{B_\rho} D_n\varphi_{E_t}\end{pmatrix} = \lim_{t\to 0}\frac{\int_{B_\rho} D\varphi_{E_t}}{\int_{B_\rho}|D\varphi_{E_t}|} = \lim_{t\to 0}\frac{\int_{B_{t\rho}} D\varphi_E}{\int_{B_{t\rho}}|D\varphi_E|} = \nu(0) = \begin{pmatrix}-1\\ 0\\ \vdots\\ 0\end{pmatrix} \tag{2.18}$$

Second, one make an immediate observation that for each $\rho > 0$, $\{\varphi_{E_t}\}_t \subset BV(B_\rho)$ because E is Caccioppoli, and for each t, $B_{t\rho}$ is bounded, hence $\varphi_E \in BV(B_{t\rho})$ and RHS of (2.17) is bounded. Again, since $0 \in \partial^* E$, one has uniform density estimate. Applying second part of (2.11), together with (2.17) yields

$$\limsup_{t\to 0} \int_{B_\rho} |D\varphi_{E_t}| = \limsup_{t\to 0} \frac{1}{t^{n-1}} \int_{B_{t\rho}} |D\varphi_E| \le C < \infty \tag{2.19}$$

Hence the sequence of functions $\{\varphi_{E_t}\}$ is uniformly bounded in $BV(B_\rho)$ norm for each $\rho > 0$. Thus by compactness theorem 1.1.4, there exists a subsequence $\{\varphi_{E_j}\}$ where $E_j := E_{t_j}$ s.t. $\varphi_{E_j} \to f$ in $L^1_{loc}(\mathbb{R}^n)$ (by unique limit on each ball B_ρ) and that $f \in BV(\mathbb{R}^n)$. Since f is L^1 limit of characteristic functions, $f = \varphi_C$ for some Borel set $C \subset \mathbb{R}^n$. Since $\varphi_C \in BV(\mathbb{R}^n)$, indeed C is Caccioppoli. Moreover, by De La Vallée Poussin Theorem, for a.e. ρ s.t. $\int_{\partial B_\rho} |D\varphi_C| = 0$, one has approximation in vector-valued radon measure

$$\lim_{t_j\to 0} \int_{B_\rho} D\varphi_{E_j} = \int_{B_\rho} D\varphi_C \tag{2.20}$$

hence combining with (2.18) gives, for the x_1 direction

$$\lim_{t_j\to 0} \int_{B_\rho} |D\varphi_{E_j}| = -\lim_{t_j\to 0} \int_{B_\rho} D_1\varphi_{E_j} = -\int_{B_\rho} D_1\varphi_C$$

Now since $\varphi_{E_j} \to \varphi_C$ in $L^1_{loc}(\mathbb{R}^n)$, by semicontinuity 1.1.1

$$\int_{B_\rho} |D\varphi_C| \le \lim_{t_j\to 0} \int_{B_\rho} |D\varphi_{E_j}| = -\int_{B_\rho} D_1\varphi_C \tag{2.21}$$

but since any other $\int_{B_\rho} D_i\varphi_C = 0$ for $i \ge 2$ as in (2.18), the equality in (2.21) holds. Now by Lebesgue-Besicovitch Differentiation 2.1.2

$$D_1\varphi_C = \left(\lim_{t\to 0} \frac{\int_{B_\rho} D_1\varphi_C}{\int_{B_\rho} |D\varphi_C|}\right) |D\varphi_C| = -|D\varphi_C| \qquad \textit{on all Borel sets}$$

$$D\varphi_C = \left(\lim_{t\to 0} \frac{\int_{B_\rho} D\varphi_C}{\int_{B_\rho} |D\varphi_C|}\right) |D\varphi_C| = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} |D\varphi_C| \qquad \textit{on all Borel sets}$$

Hence $D_i\varphi_C = 0$ as Borel measure for $i \ge 2$. Therefore φ_C depends only on x_1 and $D_1\varphi_C < 0$ implies φ_C is non-increasing in x_1. Thus $C = \{x \in \mathbb{R}^n \mid x_1 < \lambda\}$ a.e. for some $\lambda \in \mathbb{R}$. One wish to determine λ. Suppose $\lambda < 0$, then we may construct ball $B_{|\lambda|}$ around 0 that does not intersect C, so using $\varphi_{E_j} \to \varphi_C$ in $L^1_{loc}(\mathbb{R}^n)$

$$\begin{aligned} 0 = |C \cap B_{|\lambda|}| &= \int_{B_{|\lambda|}} \varphi_C(x)\,dx = \lim_{t_j\to 0} \int_{B_{|\lambda|}} \varphi_{E_j}(x)\,dx \\ &= \lim_{t_j\to 0} \frac{1}{t_j^n} \int_{B_{|\lambda|}} \varphi_E(t_j x)\,d(t_j x) = \lim_{t_j\to 0} \frac{1}{t_j^n} \int_{B_{|\lambda| t_j}} \varphi_E(y)\,dy \\ &= \lim_{t_j\to 0} \frac{|E \cap B_{|\lambda| t_j}|}{t_j^n} \ge C_1 > 0 \end{aligned}$$

for some C_1 from (2.9), contradicting our assumption. If $\lambda > 0$, use

$$\begin{aligned} 0 = |C^c \cap B_{|\lambda|}| &= \int_{B_{|\lambda|}} \varphi_{C^c}(x)\,dx = \lim_{t_j\to 0} \int_{B_{|\lambda|}} \varphi_{E_j^c}(x)\,dx \\ &= \lim_{t_j\to 0} \frac{1}{t_j^n} \int_{B_{|\lambda| t_j}} \varphi_{E^c}(y)\,dy = \lim_{t_j\to 0} \frac{|E^c \cap B_{|\lambda| t_j}|}{t_j^n} \ge C_2 > 0 \end{aligned}$$

for some C_2 from (2.10). Hence $\lambda = 0$, and so $C = T^+ = \{x \in \mathbb{R}^n \mid x_1 < 0\}$ a.e. It remains to show for any open set $A \subset \mathbb{R}^n$ s.t. $H_{n-1}(\partial A \cap T(0)) = 0$, (2.15) holds. First note that, since T^+ has smooth boundary, one use remark 1.1.1 so that $|D\varphi_{T^+}| = H_{n-1} \llcorner \partial T^+ = H_{n-1} \llcorner T(0)$ as Borel measures. So if $H_{n-1}(\partial A \cap T(0)) = 0$ for some A open, in fact $\int_{\partial A} |D\varphi_{T^+}| = 0$. But this is condition for (1.8) where the equality in semicontinuity holds in subdomains. Hence apply (1.8), one directly arrives at (2.15).

□

Corollary 2.1.1 (Density Estimates on single side of Tangent Plane to Reduced Boundary)**.** *Let* $E \subset \mathbb{R}^n$ *Caccioppoli, and* $0 \in \partial^* E$ *with* $\nu(0) = (-1, 0, \cdots, 0)$*. Then the volumne density on single side vanishes*

$$\lim_{\rho\to 0} \frac{1}{\rho^n} |E \cap B_\rho \cap T^-| = 0 \tag{2.22}$$

$$\lim_{\rho\to 0} \frac{1}{\rho^n} |(B_\rho \setminus E) \cap T^+| = 0 \tag{2.23}$$

and for any $\rho, \varepsilon > 0$*, denoting*

$$S_{\rho,\varepsilon} := B_\rho \cap \{x \in \mathbb{R}^n \mid |\langle \nu(0), x\rangle| < \varepsilon\rho\} = B_\rho \cap \{x \in \mathbb{R}^n \mid |x_1| < \varepsilon\rho\}$$

the perimeter density takes up constant portion for any $\varepsilon > 0$

$$\lim_{\rho\to 0} \frac{1}{\rho^{n-1}} \int_{S_{\rho,\varepsilon}} |D\varphi_E| = \omega_{n-1} \tag{2.24}$$

where ω_{n-1} *is volumne of* $n-1$*-dim unit ball.*

Proof. Under definition (2.14), $T^+_\rho = T^+$ and $T^-_\rho = T^-$ for any $\rho > 0$. By change of variables as in (2.16)

$$\begin{aligned}
\frac{1}{\rho^n}|E \cap B_\rho \cap T^-| &= \frac{1}{\rho^n}\int_{B_\rho} \varphi_E(x)\varphi_{T^-}(x)\, dx = \int_{B_1} \varphi_E(\rho y)\, \varphi_{T^-}(\rho y)\, dy \\
&= \int_{B_1} \varphi_{E_\rho}(y)\, \varphi_{T^-_\rho}(y)\, dy = |E_\rho \cap B_1 \cap T^-| \\
\frac{1}{\rho^n}|(B_\rho \setminus E) \cap T^+| &= \frac{1}{\rho^n}\int_{B_\rho} \varphi_{E^c}(x)\, \varphi_{T^+}(x)\, dx = \int_{B_1} \varphi_{E^c}(\rho y)\, \varphi_{T^+}(\rho y)\, dy \\
&= \int_{B_1} \varphi_{E^c_\rho}(y)\, \varphi_{T^+_\rho}(y)\, dy = |(B_1 \setminus E_\rho) \cap T^+|
\end{aligned}$$

But from Theorem 2.1.2, $E_\rho \to T^+$ in $L^1_{loc}(\mathbb{R}^n)$ up to a subsequence, hence

$$\lim_{\rho\to 0} \frac{1}{\rho^n}|E \cap B_\rho \cap T^-| = \lim_{\rho\to 0} |E_\rho \cap B_1 \cap T^-| = |T^+ \cap B_1 \cap T^-| = 0$$

$$\lim_{\rho\to 0} \frac{1}{\rho^n}|(B_\rho \setminus E) \cap T^+| = \lim_{\rho\to 0} |(B_1 \setminus E_\rho) \cap T^+| = |(B_1 \setminus T^+) \cap T^+| = 0$$

so (2.22) and (2.23) hold. Moreover, by the exact same procedure with $S_{\rho,\varepsilon}$ in place of B_ρ and $S_{1,\varepsilon}$ in place of B_1 as in (2.16), one has

$$\frac{1}{\rho^{n-1}} \int_{S_{\rho,\varepsilon}} |D\varphi_E| = \int_{S_{1,\varepsilon}} |D\varphi_{E_\rho}|$$

and since $S_{1,\varepsilon}$ is open set with $H_{n-1}(\partial S_{1,\varepsilon} \cap T) = 0$, apply (2.15) to conclude (2.24)

$$\lim_{\rho\to 0} \frac{1}{\rho^{n-1}} \int_{S_{\rho,\varepsilon}} |D\varphi_E| = \lim_{\rho\to 0} \int_{S_{1,\varepsilon}} |D\varphi_{E_\rho}| = H_{n-1}(T \cap S_{1,\varepsilon}) = \omega_{n-1}$$

□

2.2 Regularity of Reduced Boundary

The purpose of this section is to argue that for $E \subset \mathbb{R}^n$ Caccioppoli

- $\partial^* E$ is countable union of C^1 hypersurfaces up to set of $|D\varphi_E|$-measure zero.
- $\partial^* E$ is dense in ∂E.
- $\int_\Omega |D\varphi_E| = H_{n-1}(\partial^* E \cap \Omega)$ so $|D\varphi_E| = H_{n-1} \llcorner \partial^* E$ as Radon measures.

One shall first recall the precise definition for Hausdorff measure.

Definition 2.2.1. *Let* $A \subset \mathbb{R}^n$*,* $0 \le k < \infty$ *and* $0 < \delta \le \infty$*. We define the k-dim Hausdorff outer measure at step* δ

$$H^\delta_k(A) := \frac{\omega_k}{2^k} \inf \left\{ \sum_{j=1}^\infty \operatorname{diam}(S_j)^k \mid A \subset \bigcup_{j=1}^\infty S_j,\ \operatorname{diam}(S_j) < \delta\ \forall j \right\} \tag{2.25}$$

and consequently define

$$H_k(A) := \lim_{\delta\to 0} H^\delta_k(A) = \sup_{0<\delta\le\infty} H^\delta_k(A)$$

as k-dim Hausdorff measure. Here $\omega_k := \Gamma(\frac{1}{2})^k / \Gamma(\frac{k}{2}+1)$ *for* $k \ge 0$ *is measure of unit ball in* $\mathbb{R}^k$*.*

Lemma 2.2.1 (Ratio Estimate). *$E \subset \mathbb{R}^n$ Caccioppoli. $B \subset \partial^* E$. Then*

$$H_{n-1}(B) \leq 2 \cdot 3^{n-1} \int_B |D\varphi_E| \tag{2.26}$$

Proof. Since $|D\varphi_E|$ is Radon measure on $\mathbb{R}^n$, it can be approximated from the outside by open sets. Given B, for any $\eta > 0$, there exists $B \subset A$ open s.t.

$$\int_A |D\varphi_E| \leq \int_B |D\varphi_E| + \eta \tag{2.27}$$

Moreover, for any $\varepsilon > 0$, apply (2.24) to arbitrary $x \in B$, there exists $0 < \rho(x) < \varepsilon$ s.t.

$$B_{\rho(x)}(x) \subset A \qquad \textit{and} \qquad \int_{B_{\rho(x)}(x)} |D\varphi_E| \geq \frac{1}{2}\rho(x)^{n-1}\omega_{n-1} \tag{2.28}$$

One think about covering B using balls $\{B_{\rho(x)}(x)\}$ via lemma 1.2.3. So there exists $\{x_i\} \subset B$ s.t.

$$B \subset \bigcup_{i=1}^{\infty} B_{3\rho(x_i)}(x_i) \qquad \textit{and} \qquad B_{\rho(x_i)}(x_i) \cap B_{\rho(x_j)}(x_j) = \varnothing \textit{ for } i \neq j$$

and (2.28) holds for each x_i. Hence one may bound, using $B_{\rho(x_i)}(x_i) \subset A$ and disjoint, and then (2.27)

$$\begin{aligned}\sum_{i=1}^{\infty}(3\rho(x_i))^{n-1} &\leq \sum_{i=1}^{\infty} 3^{n-1}\frac{2}{\omega^{n-1}}\int_{B_{\rho(x_i)}(x_i)} |D\varphi_E| \leq \frac{2 \cdot 3^{n-1}}{\omega^{n-1}}\int_A |D\varphi_E| \\ &\leq \frac{2 \cdot 3^{n-1}}{\omega^{n-1}}\left(\int_B |D\varphi_E| + \eta\right)\end{aligned}$$

Hence recalling (2.25), since $B \subset \bigcup_{i=1}^{\infty} B_{3\rho(x_i)}(x_i)$ with $\rho(x_i) < \varepsilon$ universal bound in i

$$H_{n-1}(B) \leq \lim_{\varepsilon \to 0} \frac{\omega_{n-1}}{2^{n-1}} \inf\left\{\sum_{i=1}^{\infty}(2 \cdot 3\rho(x_i))^{n-1} \mid \rho(x_i) < \varepsilon\right\} \leq 2 \cdot 3^{n-1}\left(\int_B |D\varphi_E| + \eta\right)$$

take $\eta \to 0$ to conclude (2.26). $\square$