Solutions to Evans SDE

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39 Let $u = u(x,t)$ be a smooth solution of the *backwards diffusion equation*

$$u_t + \frac{1}{2}u_{xx} = 0$$

Let $W(\cdot)$ be one-dimensional Brownian motion. Show that for each time $t > 0$, $\mathbb{E}(u(W(t),t)) = u(0,0)$.
Answer: Note by Itô formula, we have $du(X,t) = u_t dt + u_x dX + \frac{1}{2}u_{xx} dX dX$, where in our case, $X = W(t)$, so $dX = 1dW$. Since Brownin Motion has quadratic variation, i.e., $dWdW = dt$, our formula reduces to $du(W,t) = (u_t + \frac{1}{2}u_{xx})dt + u_x dW$. Notice u satisfies backwards diffusion equation, so $du(W,t) = u_x dW$. We integrate on both sides from 0 to t to obtain $u(W,t) - u(0,0) = \int_0^t u_x dW$. Since $u(X,t)$ is smooth over t, in particular, for any fixed time t, $u \in \mathcal{L}^2(0,t)$. Then we have $\mathbb{E}(\int_0^t u_x dW) = 0$ by theorem from 4.2.3. Applying expectation on both sides gives $\mathbb{E}(u(W(t),t)) = u(0,0)$.

40 Calculate $\mathbb{E}(B^2(t))$ for the Brownian Bridge $B(\cdot)$, and show in particular that $\mathbb{E}(B^2(t)) \to 0$ as $t \to 1^-$.
Answer: Recall example 4 from 5.1.2. the solution to the Brownian Bridge Initial Value problem

$$\begin{cases} dB = -\frac{B}{1-t}dt + dW \quad (0 \le t < 1) \\ B(0) = 0 \end{cases}$$

has the form

$$B(t) = (1-t)\int_0^t \frac{1}{1-s}dW \quad (0 \le t < 1)$$

We calculate

$$\mathbb{E}\, B^2 = (1-t)^2\, \mathbb{E}(\textstyle\int_0^t \frac{1}{1-s}dW)^2 = (1-t)^2\, \mathbb{E}(\int_0^t \frac{1}{(1-s)^2}ds)$$

The second equality holds for Itô Isometry from 4.2.3. Then we apply a change of variable $\eta = 1 - s$ to obtain

$$\mathbb{E}\, B^2 = (1-t)^2 \int_{1-t}^1 \frac{1}{\eta^2}d\eta = (1-t)^2 \left(-\frac{1}{\eta}\bigg|_{1-t}^1 \right) = (1-t)^2 \left(\frac{1}{1-t} - 1 \right) = 1 - t - (1-t)^2$$

Thus as $t \to 1^-$, $\mathbb{E}(B^2(t)) \to 0$.

41 Let X solve the *Langevin Equation*, and $X_0 \sim \mathcal{N}(0, \frac{\sigma^2}{2b})$. Show that $\mathbb{E}(X(s)X(t)) = \frac{\sigma^2}{2b}e^{-b|t-s|}$.
Answer: Recall example 5 from 5.1.2. the solution to the *Langevin Equation* Initial Value problem

$$\begin{cases} dX = -bXdt + \sigma dW \\ X(0) = X_0 \end{cases}$$

for $b > 0$ friction coefficient and $\sigma \in \mathbb{R}$ diffusion coefficient has the form

$$X(t) = e^{-bt}X_0 + \sigma\int_0^t e^{-b(t-\tau)}dW(\tau) \quad (t \ge 0)$$

We calculate explicitly, WLOG, for $s > t \geq 0$

$$\begin{array}{l} X(s)X(t) = e^{-b(s+t)}X_0^2 + \sigma e^{-bs}X_0 \int_0^t e^{-b(t-\tau)}dW(\tau) \\ +\sigma e^{-bt}X_0 \int_0^s e^{-b(s-\tau)}dW(\tau) + \sigma^2 \int_0^s e^{-b(s-\tau)}dW(\tau) \int_0^t e^{-b(t-\tau)}dW(\tau) \end{array}$$

Our trick to compute the expectation is to rewrite $\int_0^t e^{-b(t-\tau)}dW(\tau) = \int_0^s \chi_{[0,t]}e^{-b(t-\tau)}dW(\tau)$ and apply (iv) from 4.2.3. Note that expectation for the two integrals in the middle equal 0 by (ii) from 4.2.3.

$$\begin{aligned} \mathbb{E}(X(s)X(t)) &= e^{-b(s+t)}\,\mathbb{E}(X_0^2) + \sigma^2\,\mathbb{E}\left(\int_0^s e^{-b(s-\tau)}dW(\tau) \int_0^s \chi_{[0,t]}e^{-b(t-\tau)}dW(\tau)\right) \\ &= e^{-b(s+t)} \cdot \frac{\sigma^2}{2b} + \sigma^2\,\mathbb{E}\left(\int_0^s \chi_{[0,t]}e^{-b(s+t-2\tau)}d\tau\right) \\ &= e^{-b(s+t)}\left(\frac{\sigma^2}{2b} + \sigma^2 \int_0^t e^{2b\tau}d\tau\right) \\ &= e^{-b(s+t)}\left[\frac{\sigma^2}{2b} + \frac{\sigma^2}{2b}\left(e^{2bt} - 1\right)\right] \\ &= \frac{\sigma^2}{2b}e^{-b|t-s|} \end{aligned}$$

The last equality holds by our choice $s > t \geq 0$.

42 (i) Consider the ODE

$$\begin{cases} \dot{x} = x^2 \quad (t > 0) \\ x(0) = x_0 \end{cases}$$

Show that if $x_0 > 0$, the solution blows up to infinity in finite time.

Answer: Recall *Local Existence and Uniqueness Theory of* 1*st Order Non-linear ODE*, as x^2 and $\frac{\partial}{\partial x}x^2 = 2x$ are both defined and continuous around $t = 0$, we know there exists unique local solution to the Initial Value Problem. Then we directly calculate the separable ODE formally

$$\int x^{-2}dx = \int 1dt$$

so that we obtain $-x^{-1} = t + C$. Plugging in the initial value we arrive at $C = -x_0^{-1}$, which makes sense as $x_0 > 0$. Therefore the local solution has the form $x(t) = -\frac{x_0}{x_0 t - 1}$ for some finite interval around $t = 0$. Now clearly as t approach x_0^{-1} from the left, the solution explodes to infinity.

(ii) Next, look at the ODE

$$\begin{cases} \dot{x} = x^{\frac{1}{2}} \quad (t > 0) \\ x(0) = 0 \end{cases}$$

Show that this problem has infinitely many nonnegative solutions.

Answer: It is obvious that $x \equiv 0$ is a solution. We calculate the separable ODE formally

$$\int x^{-\frac{1}{2}}dx = \int 1dt$$

and obtain $2x^{\frac{1}{2}} = t + C$. Plugging in $x(0) = 0$ we have $C = 0$, and thus $x(t) = \frac{t^2}{4}$ is also a solution to the ODE. The problem now lies in how we combine the above 2 solutions to generate infinitely many solutions. We implement the idea of right translation to the solution $x(t) = \frac{t^2}{4}$. To do so, we define $x_C = \frac{(t-C)^2}{4}$ for any $C \geq 0$, and observe that $\dot{x_C} = \frac{t-C}{2} = x_C^{\frac{1}{2}}$ for any $t \geq C$. We then can paste the left with $x \equiv 0$ and define a new solution $x_C = \begin{cases} \frac{(t-C)^2}{4} & (t \geq C) \\ 0 & (0 \leq t < C) \end{cases}$ We observe this is solution to our original Initial Value Problem. But $C \geq 0$ is arbitrary, so we have infinitely many nonnegative solutions.

43 (i) Use the substitution $X = u(W)$ to solve the SDE

$$\begin{cases} dX = -\frac{1}{2}e^{-2X}dt + e^{-X}dW \\ X(0) = x_0 \end{cases}$$

Answer: Since $u(W) = X$, we have $u_t = 0$. Plugging into Itô formula, we have

$$dX = u_x(W)dW + \frac{1}{2}u_{xx}(W)dt$$

Thus we equate $u_x(W) = e^{-X} = e^{-u(W)}$. We solve the separable ODE formally

$$\int e^u du = \int 1 dx$$

and obtain $e^u = x + C$. Plugging in initial value $u(0) = X(0) = x_0$, we have $e^{u(x)} = x + e^{x_0}$. Then taking logarithm on both sides, we have

$$u(x) = \log(x + e^{x_0})$$

Plugging in $x = W$, we arrive at

$$X = u(W) = \log(W + e^{x_0})$$

We check our calculation by seeing

$$-\frac{1}{2}e^{-2X} = -\frac{1}{2\left(W + e^{x_0}\right)^2} = \frac{1}{2}u_{xx}(W)$$

(ii) Show that the solution blows up at a finite, random time.
Answer: By our explicit solution $X = u(W) = \log(W + e^{x_0})$, it suffices to show that W reaches $-e^{x_0}$ for some finite time almost surely. We know for simple random walk $S_n = \sum_{i=1}^n X_i$ where X_i are i.i.d. Bernoulli Random Variables, we have for any $C < \infty$, $\mathbb{P}\{\inf\{n \geq 0 |\ S_n = C\} < \infty\} = 1$. Then for Brownian Motion W as limit of S_n by construction, it indeed inherits the property and has $\mathbb{P}\{\inf\{t \geq 0 |\ W(t) = C\} < \infty\} = 1$. Take $C = -e^{x_0}$.

44 Solve the SDE $dX = -Xdt + e^{-t}dW$
Answer: Note that this is Linear SDE in the narrow sense. We directly apply Theorem 5.4.2 (i) choosing $\mathbf{c} \equiv 0$, $\mathbf{D} \equiv -1$, and $\mathbf{E}(t) = e^{-t}$, with arbitrary initial condition $X(0) = X_0$. We have the solution in explicit form

$$X(t) = e^{-t}X_0 + \int_0^t e^{-(t-s)}e^{-s}dW = e^{-t}X_0 + e^{-t}W(t)$$

45 Let $\mathbf{W} = \left(W^1, W^2, \cdots, W^n\right)$ be n-dimensional Brownian Motion and write

$$\mathcal{R} := |\mathbf{W}| = \left(\sum_{i=1}^n (W^i)^2\right)^{\frac{1}{2}}$$

Show that $\mathcal{R}$ solves the *Stochastic Bessel Equation*

$$d\mathcal{R} = \frac{n-1}{2\mathcal{R}}dt + \sum_{i=1}^n \frac{W^i}{\mathcal{R}}dW^i$$

Answer: We write $\mathcal{R} = u(\mathbf{W},t) = u(W^1,\cdots,W^n,t) = \left(\sum_{i=1}^n (W^i)^2\right)^{\frac{1}{2}}$. We immediately have $u_t = 0$ and for any $i,j \in \{1,\cdots,n\}$

$$u_{x_i} = \frac{W^i}{\mathcal{R}}, \quad u_{x_i x_j} = \frac{\delta_{ij}}{\mathcal{R}} - \frac{W^i W^j}{\mathcal{R}^3}$$

for $\delta_{ij} = \begin{cases} 1 & (i=j) \\ 0 & (i \neq j) \end{cases}$ Also Note $d\mathbf{W} = \mathbf{G}d\mathbf{W}$ for $\mathbf{G} = \mathbf{I}_n$ the $n \times n$ identity matrix. Then by Itô's chain rule in dimension n from 4.4.2. we have

$$du(\mathbf{W},t) = \sum_{i=1}^n u_{x_i} dW^i + \frac{1}{2}\sum_{i=1}^n u_{x_i x_i} dt = \sum_{i=1}^n \frac{W^i}{\mathcal{R}} dW^i + \frac{1}{2}\sum_{i=1}^n \left(\frac{1}{\mathcal{R}} - \frac{(W^i)^2}{\mathcal{R}^3}\right) dt$$

We conclude by observing that $\sum_{i=1}^n \frac{(W^i)^2}{\mathcal{R}^3} = \frac{1}{\mathcal{R}}$.

46 (i) Show that $\mathbf{X} = (\cos(W), \sin(W))$ solves the system of SDE

$$\begin{cases} dX^1 = -\frac{1}{2}X^1 dt - X^2 dW \\ dX^2 = -\frac{1}{2}X^2 dt + X^1 dW \end{cases}$$

Answer: By direct computation

$$d\cos(W) = -\sin(W)dW - \frac{1}{2}\cos(W)dt$$

$$d\sin(W) = \cos(W)dW - \frac{1}{2}\sin(W)dt$$

(ii) Show also that if $\mathbf{X} = (X^1, X^2)$ is any other solution, then $|\mathbf{X}|$ is constant in time.
Answer: It suffices to show that $d\left|\mathbf{X}\right|^2 = d\left((X^1)^2 + (X^2)^2\right) = 0$. Notice

$$d((X^1)^2) = 2X^1 dX^1 + (dX^1)^2 = 2X^1 dX^1 + (X^2)^2 dt = \left((X^2)^2 - (X^1)^2\right) dt - 2X^1 X^2 dW$$

$$d((X^2)^2) = 2X^2 dX^2 + (dX^2)^2 = 2X^2 dX^2 + (X^1)^2 dt = \left((X^1)^2 - (X^2)^2\right) dt + 2X^1 X^2 dW$$

Summing up gives $d\left((X^1)^2 + (X^2)^2\right) = 0$. Note we used the formal cancellation rule from 4.4.3. that

$$\begin{cases} (dt)^2 = 0 \\ dt dW = 0 \\ (dW)^2 = dt \end{cases}$$

47 Solve the system of SDE

$$\begin{cases} dX^1 = dt + dW^1 \\ dX^2 = X^1 dW^2 \end{cases}$$

where $\mathbf{W} = \left(W^1, W^2\right)$ is 2-dimensional Brownian Motion.
Answer: We integral w.r.t. to the first equation to obtain $X^1 = t + W^1 + X^1_0$ for some initial value $X^1(0) = X^1_0$. Then plugging directly into the second equation, we get

$$dX^2 = t dW^2 + W^1 dW^2 + X^1_0 dW^2 = d(tW^2) - W^2 dt + W^1 dW^2 + X^1_0 dW^2$$

Thus integrating w.r.t. the second equation we have

$$X^2 = tW^2 - \int_0^t W^2 dt + \int_0^t W^1 dW^2 + X^1_0 W^2 + X^2_0$$

for some initial value $X^2(0) = X^2_0$.

48 Solve the system of SDE

$$\begin{cases} dX^1 = X^2 dt + dW^1 \\ dX^2 = X^1 dt + dW^2 \end{cases}$$

where $\mathbf{W} = (W^1, W^2)$ is 2-dimensional Brownian Motion.

Answer: Note that this is Linear system of SDE in the narrow sense. We directly apply Theorem 5.4.2 (i) choosing $\mathbf{c} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathbf{D} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\mathbf{E} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, with arbitrary initial condition $\mathbf{X}(0) = \mathbf{X_0}$. We have the solution in explicit form

$$\mathbf{X}(t) = e^{\mathbf{D}t}\mathbf{X}_0 + \int_0^t e^{\mathbf{D}(t-s)} \begin{pmatrix} dW^1 \\ dW^2 \end{pmatrix}$$

where the matrix exponential is defined as $e^{\mathbf{D}t} := \sum_{k=0}^{\infty} \frac{\mathbf{D}^k t^k}{k!}$. In fact we can calculate the exponential explicitly. First we find for matrix $\mathbf{D} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ its eigenvalues $\lambda_1 = 1, \lambda_2 = -1$ and the corresponding eigenvectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, then diagonalize $\mathbf{D} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$, where $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$. Thus $e^{\mathbf{D}t} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-1} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}$ for $\cosh(t) = \frac{1}{2}(e^t + e^{-t})$ and $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$. Thus, for arbitrary initial condition $\mathbf{X}(0) = \begin{pmatrix} \mathbf{X_0}^1 \\ \mathbf{X_0}^2 \end{pmatrix}$,

$$\mathbf{X}(t) = \begin{pmatrix} \cosh(t)\mathbf{X_0}^1 + \sinh(t)\mathbf{X_0}^2 \\ \sinh(t)\mathbf{X_0}^1 + \cosh(t)\mathbf{X_0}^2 \end{pmatrix} + \int_0^t \begin{pmatrix} \cosh(t-s)dW^1 + \sinh(t-s)dW^2 \\ \sinh(t-s)dW^1 + \cosh(t-s)dW^2 \end{pmatrix}$$

49 Solve

$$\begin{cases} dX = \frac{1}{2}\sigma'(X)\sigma(X)dt + \sigma(X)dW \\ X(0) = 0 \end{cases}$$

where W is one-dimensional Brownian Motion and σ is a smooth, positive function.

Answer: We define $f(x) := \int_0^x \frac{dy}{\sigma(y)}$ and set $g := f^{-1}$. The inverse function g makes sense because σ is positive function, so f is strictly increasing. Then notice

$$g'(W) = \left(f^{-1}\right)'(W) = \frac{1}{\frac{1}{\sigma(f^{-1}(W))}} = \sigma\left(f^{-1}(W)\right) = \sigma\left(g(W)\right)$$

$$g''(W) = \sigma'\left(g(W)\right) g'(W) = \sigma'\left(g(W)\right)\sigma\left(g(W)\right)$$

Thus we observe by Itô formula

$$dg(W) = g'(W)dW + \frac{1}{2}g''(W)dt = \sigma\left(g(W)\right)dW + \frac{1}{2}\sigma'\left(g(W)\right)\sigma\left(g(W)\right)dt$$

We notice $g(W(0)) = g(0) = f^{-1}(0) = 0$, so $X = g(W)$ is indeed a solution to our problem. Finally, we check that our problem satisfies the condition for Theorem 5.2.3. as σ is smooth, so we admit unique solution $X = g(W)$ over certain time interval.

50 Let τ be the first time a one-dimensional Brownian Motion $W(\cdot)$ hits the half-open interval $(a, b]$. Show that τ is a stopping time.

Answer: Note $\tau = \inf\{t \geq 0 | \, W(t) \in (a, b]\}$. We define our probability space $(\Omega, \mathcal{U}, \mathbb{P})$ and our filtration $\mathcal{F}(\cdot)$ w.r.t. the Brownian Motion $W(\cdot)$. Recall that a random variable is a stopping time w.r.t. filtration $\mathcal{F}(\cdot)$ if $\{\tau \le t\} \in \mathcal{F}(t)$ for all $t \ge 0$. We take a countable dense subset $\{t_i\}_{i\in\mathbb{N}} \subseteq [0,\infty)$, and rewrite

$$\{\tau < t\} = \bigcup_{t_i<t} \{W(t_i) \in (a,b]\} = \bigcap_{n=1}^{\infty} \bigcup_{t_i<t} \{W(t_i) \in \left(a, b + \frac{1}{n}\right)\}$$

But by definition of filtration $\mathcal{F}(\cdot)$, we have

$$\mathcal{F}(t) \supseteq \mathcal{W}(t) := \mathcal{U}\left(W(s) | 0 \le s \le t\right) \quad \forall t \ge 0$$

Thus $\{W(t_i) \in \left(a, b + \frac{1}{n}\right)\} \in \mathcal{F}(t_i) \subseteq \mathcal{F}(t)$ for any $t_i < t$, and so $\{\tau < t\} \in \mathcal{F}(t)$. Finally notice

$$\{\tau \le t\} = \bigcap_{n=1}^{\infty} \{\tau < t + \frac{1}{n}\} \in \bigcap_{n=1}^{\infty} \mathcal{F}(t + \frac{1}{n}) = \mathcal{F}(t)$$

where the last inequality holds by assumption. This concludes our proof.

51 Let $\mathbf{W}$ denote n-dimensional Brownian Motion for $n \ge 3$. Let $\mathbf{X} = \mathbf{W} + x_0$ where $x_o \in U = \{0 < R_1 < |x| < R_2\}$. Calculate explicitly the probability that $\mathbf{X}$ will hit the outer sphere $\{|x| = R_2\}$ before hitting the inner sphere $\{|x| = R_1\}$.

Answer: We apply Example 3 from 6.2.1. defining the two disjoint regions $\Gamma_1 = \{|x| = R_2\}$ and $\Gamma_2 = \{|x| = R_1\}$. They we construct boundary value problem for Laplace Equation

$$\begin{cases} \Delta u = 0 & in\ U = \{0 < R_1 < |x| < R_2\} \\ u = 1 & on\ \Gamma_1 = \{|x| = R_2\} \\ u = 0 & on\ \Gamma_2 = \{|x| = R_1\} \end{cases}$$

and we know the explicit solution u is the probability of $\mathbf{X}$ hitting Γ_1 before hitting Γ_2. We building the solution $u = \frac{\Phi(x) - \Phi(R_1)}{\Phi(R_2) - \Phi(R_1)}$ where $\Phi(x) := |x|^{2-n}$, since $\Phi(x) := |x|^{2-n}$ is harmonic function and our u satisfies the boundary values.