MATH-SHU 997 Independent Study Reflection

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1 Overview

Throughout Fall 2022, I read through whole Chapter 2 of Evans's Textbook: *Partial Differential Equations*, and a brief overview at Chapter 9 of Arnold's Textbook: *Mathematical Methods of Classical Mechanics*. We manage to have bi-weekly discussions, with a focus on Laplace, Heat, and Wave Equations. In particular, the topics covered are listed below:

1. Transport Equation. Initial-value problem. Nonhomogeneous Initial-value problem.
2. Laplace Equation. Fundamental Solutions and solution to Poisson's Equation. Mean Value Formulas. Properties of Harmonic Functions. Green's Function. Energy Methods.
3. Heat Equation. Fundamental Solutions, Initial-value problem and Nonhomogeneous problem. Mean-value formula. Properties of Solutions. Energy Methods.
4. Wave Equation. Solution by spherical means. Nonhomogeneous problem. Energy Methods.

2 Evans Partial Differential Equations

2.1 Transport Equation

We solve the general solution to the transport equation $u_t + b \cdot Du = 0$ in $\mathbb{R}^n \times (0, \infty)$ by constructing $z(s) = u(x + sb, t + s)$ for $s \in \mathbb{R}$. We observe that $\frac{d}{ds}z = 0$, so u remains constant on the characteristic line through (x, t) along the direction $(b, 1)$.
In particular, we solve the same equation with initial value $u = g$ constraint on $\mathbb{R}^n \times \{t = 0\}$ by observing $z(-t) = u(x-tb, 0) = g(x-tb)$. Since u is constant along each characteristic line with direction $(b, 1)$, and g gives all values of u at the intercept of each line with the plane $\mathbb{R}^n \times \{t = 0\}$, we construct solution $u(x, t) = g(x - tb)$ $\forall t \geq 0$ provided g is C^1.
For **Nonhomogeneous Initial-value problem**, we solve for $u_t + b \cdot Du = f$ in $\mathbb{R}^n \times (0, \infty)$ with initial value constaint $u = g$ on $\mathbb{R}^n \times \{t = 0\}$. We notice $\frac{d}{ds}z = f(x + sb, t + s)$, so by fundamental theorem of calculus, we obtain upon integration $z(0) - z(-t) = \int_{-t}^{0} f(x + sb, t + s)ds = \int_0^t f(x + (s - t)b, s)ds$, which gives $u(x, t) = g(x - tb) + \int_0^t f(x + (s - t)b, s)ds$.

2.2 Laplace Equation

We wish to study solution to Laplace Equation $\Delta u = 0$, Poisson Equation $-\Delta u = f$ and harmonic functions that satisfy Laplace Equation.

2.2.1 Fundamental Solutions

We introduce radial solutions $u(x) = v(r)$ where $r = |x| = \sqrt{\sum_{i=1}^n x_i^2}$. By direct computation, we obtain $\Delta u = \sum_{i=1}^n u_{x_i x_i} = v^{''}(r) + v^{'}(r)\frac{n-1}{r}$. So putting $v^{''}(r) + v^{'}(r)\frac{n-1}{r} = 0$ and solving for ordinary differential equation, we have general solutions in radial form $v(r) = \begin{cases} b\log(r) + c & n = 2 \\ br^{2-n} + c & n \geq 3 \end{cases}$.

We choose the set of fundamental solutions for $x \neq 0$ as $\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x| & n = 2 \\ \frac{1}{n(n-2)\alpha(n)}|x|^{2-n} & n \geq 3 \end{cases}$, where $\alpha(n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ is volume of unit ball $B(0, 1)$ in $\mathbb{R}^n$.

For **Poisson equation** $-\Delta u = f$, we construct $u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy = \begin{cases} -\frac{1}{2\pi}\int_{\mathbb{R}^n} \log(|x-y|)f(y)dy & n=2 \\ \frac{1}{n(n-2)\alpha(n)}\int_{\mathbb{R}^n} |x-y|^{2-n}f(y)dy & n\geq 3 \end{cases}$ and show that it is indeed solution and belongs to $C^2(\mathbb{R}^n)$.

2.2.2 Mean-value Formulas

We now consider open set $U \subset \mathbb{R}^n$ and let u be harmonic function within U. We establish a uniqueness characterization for harmonic functions, known as the mean-value property.

Theorem 2.1 (Mean Value Formula). *If $u \in C^2(\mathbb{R}^n)$ is harmonic, then $\forall$ ball $B(x, r) \subset U$, we have*

$$u(x) = \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(x,r)} udS = \frac{1}{\alpha(n)r^n}\int_{B(x,r)} udy \tag{1}$$

And conversely, we have the characterization for harmonic functions

Theorem 2.2. *If $\forall$ ball $B(x, r) \subset U$, $u \in C^2(\mathbb{R}^n)$ satisfies*

$$u(x) = \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(x,r)} udS \tag{2}$$

then u is harmonic in U.

2.2.3 Properties of harmonic functions

Assume open, bounded set $U \subset \mathbb{R}^n$. We first establish the maximum principles for harmonic functions, i.e., harmonic functions attain maximum only on boundary, otherwise they are constant.

Theorem 2.3 (Maximum principle). *$u \in C^2(U) \cap C(\overline{U})$ is harmonic within U, then $max_{\overline{U}}u = max_{\partial U}u$.*

Theorem 2.4 (Strong Maximum principle). *Moreover, if U is connected and exists $x_0 \in U$ s.t. $u(x_0) = max_{\overline{U}}u$ then u is constant within U.*

The maximum principles are direct results from Mean-value property. And following from maximum principles, we establish positivity and uniqueness of solutions.

Theorem 2.5 (Positivity). *If U is connected and $u \in C^2(U)\cap C(\overline{U})$ satisfies $\begin{cases} \Delta u = 0 & in \quad U \\ u = g & on \quad \partial U \end{cases}$ where $g \geq 0$, then u is positive everywhere in U if g is positive somewhere on ∂U.*

Theorem 2.6 (Uniqueness). *$g \in C(\partial U)$ and $f \in C(U)$, then there exist at most one solution $u \in C^2(U)\cap C(\overline{U})$ satisfying $\begin{cases} -\Delta u = f & in \quad U \\ u = g & on \quad \partial U \end{cases}$*

Next, we discuss the regularity theorem, where the algebraic structure of harmonic functions automatically gives infinitely differentiable. Moreover, harmonic functions are analytic.

Theorem 2.7 (Regularity). *If $u \in C(U)$ satisfies mean value property $\forall B(x,r) \subset U$, then $u \in C^\infty(U)$. (Note that u may not be smooth, or even continuous up to ∂U)*

The mean-value formulas provide pointwise local estimates on various partial derivatives of a harmonic function.

Theorem 2.8 (Estimates on derivatives). *If u is harmonic in U, then $|D^\alpha u(x_0)| \leq \frac{(2^{n+1}nk)^k}{\alpha(n)r^{n+k}}||u||_{L^1(B(x_0,r)}$ for each $B(x_0, r) \subset U$ and each multiindex α with order $|\alpha| = k$ for $k = 1, \cdots$*

With precise estimates of partial derivatives, we have analyticity.

Theorem 2.9 (Analyticity). *If u is harmonic in U, then u is analytic in U.*

Finally we see two important theorems derived from the known properties of harmonic functions. We observe harmonic functions on either $V \subset \overline{V} \subset U$ or $\mathbb{R}^n$.

Theorem 2.10 (Harnack's Inequality). *For each connected open set $V \subset \overline{V} \subset U$, there exists a positive constant C depending only on V, s.t., $sup_V u \leq C inf_V u$ for all nonnegative harmonic functions u in U. In particular, $\forall x, y \in V$, we have $\frac{1}{C}u(y) \leq u(x) \leq Cu(y)$*

Harnack's inequality asset that values of a nonnegative function u within V are all comparable. As V has a positive distance away from ∂U, there is room for averaging effects of Laplace equation. Now from another perspective, we study bounded harmonic functions on $\mathbb{R}^n$.

Theorem 2.11 (Liouville's Theorem). *If $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded, then u is constant.*

This means there are no nontrivial bounded harmonic functions on all of $\mathbb{R}^n$. In particular, if u instead solves Poisson Equation, we have a corresponding result for $n \geq 3$.

Theorem 2.12 (Representation formula). *If $f \in C^2_C(\mathbb{R}^n), n \geq 3$, then any bounded solution to $-\Delta u = f$ in $\mathbb{R}^n$ has the form $u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy + C$ for $x \in \mathbb{R}^n$ for some constant C.*

2.2.4 Green's Function

We consider open set $U \subset \mathbb{R}^n$ with ∂U is C^1. Next we obtain general representation formula for solution of Poisson Equation with prescribed boundary condition $\begin{cases} -\Delta u = f & in \quad U \\ u = g & on \quad \partial U \end{cases}$

We first derive the identity that solves the above problem with $u \in C^2(\overline{U})$

$$u(x) = \int_{\partial U} \Phi(y-x)\frac{\partial u}{\partial v}(y) - u(y)\frac{\partial \Phi}{\partial v}(y-x)dS(y) - \int_U \Phi(y-x)\Delta u(y)dy \tag{3}$$

where Φ is fundamental solution. We know values of Δu within U and u on ∂U from the construction of boundary problem, but not the value $\frac{\partial u}{\partial v}$ on ∂U. In order to remove this term, we introduce a corrector function $\phi^x = \phi^x(y)$ for each fixed x s.t. $\phi^x = \begin{cases} \Delta\phi^x = 0 & in \quad U \\ \phi^x = \Phi(y-x) & on \quad \partial U \end{cases}$, and thus equate the last term in the above identity with corrector function, i.e., $-\int_U \Phi(y-x)\Delta u(y)dy = -\int_U \phi^x(y)\Delta u(y)dy$. Then by applying Green's Formula once, we get rid of the term in the identity that involves $\frac{\partial u}{\partial v}$ on ∂U. We further introduce Green's function.

Definition 2.13. *Green's Function for region U is $G(x,y) := \Phi(y-x) - \phi^x(y)$ is for $x, y \in U, x \neq y$*

and can simply the identity into

$$u(x) = -\int_{\partial U} u(y)\frac{\partial G}{\partial v}(x,y)dS(y) - \int_U G(x,y)\Delta u(y)dy \tag{4}$$

Going back to the original problem, we develop our representation formula using Green's Function

Theorem 2.14 (Representation formula using Green's Function). *If $u \in C^2(\overline{U})$ solves $\begin{cases} -\Delta u = f & in \quad U \\ u = g & on \quad \partial U \end{cases}$, then*

$$u(x) = -\int_{\partial U} g(y)\frac{\partial G}{\partial v}(x,y)dS(y) + \int_U G(x,y)f(y)dy \qquad (x \in U)$$

Now the question lies in the construction of Green's Function for a given region U. Special cases are applicable when U has simple geometry. Before the examples, we note that Green's Function is symmetric, i.e., $G(x,y) = G(y,x)$ $\forall x, y \in U, x \neq y$.
Now we consider two simple regions, the half plane and a ball. If we have region

$$U = \mathbb{R}^n_+ := \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n | x_n > 0\} \tag{5}$$

we can define a reflection for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n_+$ w.r.t. the plane $\partial\mathbb{R}^n_+$ as $\widetilde{x} = (x_1, \cdots, x_{n-1}, -x_n)$. We build corrector function by reflecting the singularity point from $x \in \mathbb{R}^n_+$ to $\widetilde{x} \notin \mathbb{R}^n_+$ by $\phi^x(y) := \Phi(y - \widetilde{x})$ for $x, y \in \mathbb{R}^n_+$, and $\phi^x(y) := \Phi(y-x)$ for $y \in \partial\mathbb{R}^n_+$. It is indeed corrector function by definition, and so we define Green's function for the half-plane $\mathbb{R}^n_+$ as $G(x,y) = \Phi(y-x) - \Phi(y-\widetilde{x})$ for $x, y \in \mathbb{R}^n_+, x \neq y$.

We now plug in the representation formula using Green's function, and obtain the solution for Laplace equation with boundary condition $\begin{cases} \Delta u = 0 & in \quad \mathbb{R}^n_+ \\ u = g & on \quad \partial\mathbb{R}^n_+ \end{cases}$ as

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n} dy = \int_{\partial\mathbb{R}^n_+} K(x,y)g(y)dy \qquad (x \in \mathbb{R}^n_+)$$

upon defining the Poisson's kernel for $\mathbb{R}^n_+$ as $K(x,y) = \frac{2x_n}{n\alpha(n)} \frac{1}{|x-y|^n}$. The solution is known as the Poisson's formula for $\mathbb{R}^n_+$.
On the other hand, for $U = B(0,1)$, we define for $x \in \mathbb{R}^n - \{0\}$ its dual point w.r.t. $\partial B(0,1)$ as $\widetilde{x} = \frac{x}{|x|^2}$, which is inversion through unit sphere $\partial B(0,1)$. We define $\phi^x(y) := \Phi(|x|(y-\widetilde{x}))$ for $y \in \overline{B(0,1)}$. $\phi^x(y)$ is indeed corrector function since $\phi^x(y) := \Phi(y-x)$ for $y \in \partial B(0,1)$. Then we have Green's function on a unit ball as $G(x,y) := \Phi(y-x) - \Phi(|x|(y-\widetilde{x}))$ for $x, y \in B(0,1), x \neq y$. We plug in the representation formula using Green's function, and obtain the solution for Laplace equation with boundary condition $\begin{cases} \Delta u = 0 & in \quad B(0,1) \\ u = g & on \quad \partial B(0,1) \end{cases}$ as

$$u(x) = \frac{1-|x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{g(y)}{|x-y|^n} dS(y) \tag{6}$$

From here, instead of on unit ball, we can define solution on $B^0(0,r)$, the open ball with radius r, which solves the corresponding problem $\begin{cases} \Delta u = 0 & in \quad B^0(0,r) \\ u = g & on \quad \partial B(0,r) \end{cases}$. An easy approach would be defining $\widetilde{u}(x) = u(rx)$ and $\widetilde{g}(x) = g(rx)$ and replace them with the previous problem. We further obtain

$$u(x) = \frac{r^2-|x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} dS(y) = \int_{\partial B(0,r)} K(x,y)g(y)dS(y) \qquad (x \in B^0(0,r))$$

upon defining the Poisson's kernel for the ball $B^0(0,r)$ as $K(x,y) = \frac{r^2-|x|^2}{n\alpha(n)r} \frac{1}{|x-y|^n}$. The solution is known as the Poisson's formula for $B^0(0,r)$.

2.2.5 Energy Method

We look from the "energy" perspective at harmonic functions, i.e., with techniques involving the L^2-norms of various expressions. We first restate the uniqueness theorem.

Theorem 2.15 (Uniqueness with Energy Method)**.** *There exists at most one solution of $u \in C^2(\overline{U})$ that solves $\begin{cases} -\Delta u = f & in \quad U \\ u = g & on \quad \partial U \end{cases}$ for U open, bounded, and ∂U is C^1.*

Proof. Suppose $\widetilde{u}$ is another solution, define $w := u - \widetilde{u}$. We see $\Delta w = 0$ in U. Thus

$$0 = -\int_U w\Delta w dx = \int_U Dw \cdot Dw dx - \int_{\partial U} w \frac{\partial w}{\partial \upsilon} dS(x) = \int_U |Dw|^2 dx \tag{7}$$

So $|Dw| = 0$ within U. And since $w = 0$ on ∂U, we deduce $w = u - \widetilde{u}$ is constant equal to 0 in U. □

Also, the solution to $\begin{cases} -\Delta u = f & in \quad U \\ u = g & on \quad \partial U \end{cases}$ is actually minimizer of energy functional $I[w] := \int_U \frac{1}{2}|Dw|^2 - wf dx$ with $w \in \mathcal{A} := \{w \in C^2(\overline{U}) \mid w = g \quad on \ \partial U\}$.

Theorem 2.16 (Dirichlet's Principle)**.** *Let $u \in C^2(\overline{U})$ solve $\begin{cases} -\Delta u = f & in \quad U \\ u = g & on \quad \partial U \end{cases}$, then $I[u] = min_{w\in\mathcal{A}} I[w]$. Conversely, if $u \in \mathcal{A}$, then it solves the boundary-value problem.*

Essentially, if $u \in \mathcal{A}$, that u solves Poisson Equation is equivalent to u minimizes the energy I.

2.3 Heat Equation

We wish to study Heat Equation $u_t - \Delta u = 0$ and nonhomogeneous heat equation $u_t - \Delta u = f$.

2.3.1 Fundamental Solution

We look for solution of the form $u(x,t) = v(\frac{|x|^2}{t}) = v(\frac{r^2}{t})$, since $u(\lambda x, \lambda^2 t)$ and $u(x,t)$ solve the same equation $u_t - \Delta u = 0$. In particular, we look for $u(x,t) = \frac{1}{t^\alpha} v(\frac{x}{t^\beta}) = \frac{1}{t^\alpha} v(y)$ with hint that u should remain invariant under dilation scaling. It suffices to determine appropriate constants α, β and the function v to give a general solution.
We insert the expression into heat equation and if we take $\beta = \frac{1}{2}$, we reduce the expression into $\alpha v + \frac{1}{2} y \cdot Dv + \Delta v = 0$. We further simplify by taking v to be radial, i.e., $w(|y|) = v(y)$, and transform the expression into $\alpha w + \frac{1}{2} r w^{'} + w^{''} + \frac{n-1}{r} w^{'} = 0$. Now take $\alpha = \frac{n}{2}$, we solve for ordinary differential equation and derive the solution $w = be^{-\frac{r^2}{4}}$ for some constant b. We conclude that $u(x,t) = \frac{b}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$ solves Heat Equation.

We define our fundamental solution to Heat Equation as $\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & x \in \mathbb{R}^n, t > 0 \\ 0 & x \in \mathbb{R}^n, t < 0 \end{cases}$ and notice at once that for fixed $t > 0$, the fundamental solution satisfies $\int_{\mathbb{R}^n} \Phi(x,t) dx = 1$.

For the **initial-value (or Cauchy) problem** $\begin{cases} u_t - \Delta u = 0 & in \quad \mathbb{R}^n \times (0,\infty) \\ u = g & on \quad \mathbb{R}^n \times \{t = 0\} \end{cases}$, we construct solution as $u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y) dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y) dy$. We notice if $g(y)$ is bounded, continuous, non-negative and not constant 0, then u is positive for all points $x \in \mathbb{R}^n$ and $t > 0$. We interpret this observation as heat equation forces infinite propagation speed for disturbances.

For **nonhomogeneous problem** $\begin{cases} u_t - \Delta u = f & in \quad \mathbb{R}^n \times (0,\infty) \\ u = 0 & on \quad \mathbb{R}^n \times \{t = 0\} \end{cases}$ we construct our solution using *Duhamel's principle*. We first define $u(x,t;s) = \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s) ds$, and obtain our solution by integrating from 0 to t, i.e, $u(x,t) = \int_0^t u(x,t;s) ds = \int_0^t \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y,s) dy ds$ for $x \in \mathbb{R}^n$ and $t > 0$.

For **nonhomogeneous problem with general initial data** $\begin{cases} u_t - \Delta u = f & in \quad \mathbb{R}^n \times (0,\infty) \\ u = g & on \quad \mathbb{R}^n \times \{t = 0\} \end{cases}$ we combine the two previous cases and discover $u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y) dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s) dy ds$

2.3.2 Mean-value Formulas

Let $U \subset \mathbb{R}^n$ be open and bounded, fix $T > 0$. We define the parabolic cylinder as $U_T := U \times (0,T]$ and its parabolic boundary as $\Gamma_T := \overline{U}_T - U_T$.
If we regard $\partial B(x,r)$ as level sets of the fundamental solutions $\Phi(x-y)$ for Laplace equation, we construct an analogue of level set of fundamental solution $\Phi(x-y,t-s)$ for Heat Equation, known as the Heat ball.

Definition 2.17. *For fixed $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $r > 0$, we define Heat ball*

$$E(x,t;r) := \{(y,s) \in \mathbb{R}^{n+1} | s \leq t, \Phi(x-y,t-s) \geq \frac{1}{r^n}\} \tag{8}$$

We obtain the analogue for Mean-value property on Heat ball

Theorem 2.18 (Mean-value Property for Heat Equation)**.** *Let $u \in C_1^2(U_T)$ solve the heat equation, then for each $E(x,t;r) \subset U_T$, we have $u(x,t) = \frac{1}{4r^n} \iint_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dy ds$*

2.3.3 Properties of solutions

We first employ mean-value property to give maximum principles.

Theorem 2.19 (Maximum principle)**.** *Let $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ solve the heat equation in U_T, then $max_{\overline{U}_T} u = max_{\Gamma_T} u$*

Theorem 2.20 (Strong Maximum principle)**.** *Moreover, if U is connected and exists $(x_0,t_0) \in U_T$ s.t. $u(x_0,t_0) = max_{\overline{U}_T} u$, then u is constant within $\overline{U}_{t_0}$.*

This means if u attains maximum or minimum at an interior point, then u is constant at all earlier times. We immediately observe infinite propagation speed again.

Theorem 2.21 (Positivity). *If U connected and $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ solves* $\begin{cases} u_t - \Delta u = 0 & in \quad U_T \\ u = 0 & on \quad \partial U \times [0,T] \\ u = g & on \quad U \times \{t=0\} \end{cases}$ *where $g \geq 0$, then u is positive everywhere within U_T if g is positive somewhere.*

We also have uniqueness theorem as important application of maximum principle.

Theorem 2.22 (Uniqueness on bounded domains). *Let $g \in C(\Gamma_T)$, $f \in C(U_T)$, then there exists at most one solution $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ of initial/boundary-value problem* $\begin{cases} u_t - \Delta u = f & in \quad U_T \\ u = g & on \quad \Gamma_T \end{cases}$

We now extent previous theorems to the Cauchy Problem, that is, initial-value problem for $U = \mathbb{R}^n$. Since we no longer have a bounded region, we need control on the behavior of solutions for large $|x|$.

Theorem 2.23 (Maximum principle for Cauchy Problem). *Let $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solve* $\begin{cases} u_t - \Delta u = 0 & in \quad \mathbb{R}^n \times (0,T) \\ u = g & on \quad \mathbb{R}^n \times \{t=0\} \end{cases}$ *that satisfies the growth rate $u(x,t) \leq Ae^{a|x|^2}$ for $x \in \mathbb{R}^n$, $0 \leq t \leq T$ and constants $A, a > 0$, then we have $sup_{\mathbb{R}^n \times [0,T]} u = sup_{\mathbb{R}^n} g$*

Theorem 2.24 (Uniqueness for Cauchy Problem). *Let $g \in C(\mathbb{R}^n)$ and $f \in C(\mathbb{R}^n \times [0,T])$, then there exists at most one solution $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solving the initial-value problem* $\begin{cases} u_t - \Delta u = f & in \quad \mathbb{R}^n \times (0,T) \\ u = g & on \quad \mathbb{R}^n \times \{t=0\} \end{cases}$ *with growth rate $u(x,t) \leq Ae^{a|x|^2}$ for $x \in \mathbb{R}^n$, $0 \leq t \leq T$ and constants $A, a > 0$.*

Also, we derive that heat equations are automatically smooth.

Theorem 2.25 (Smoothness). *Let $u \in C_1^2(U_T)$ solve heat equation in U_T, then $u \in C^\infty(U_T)$.*

Note that the regularity assertion is valid even if u attains nonsmooth boundary values on Γ_T. In particular, we have local estimates on derivatives of solutions.

Theorem 2.26 (Estimates on derivatives). *There exists for each pais of integers $k, l = 0, \cdots$ a constant C_{kl} s.t.*

$$max_{C(x,t;\frac{r}{2})} |D_x^k D_t^l u| \leq \frac{C_{kl}}{r^{k+2l+n+2}} ||u||_{L^1(C(x,t;r))} \tag{9}$$

for all cylinders $C(x,t;\frac{r}{2}) \subset C(x,t;r) \subset U_T$ and all solutions u of heat equation in U_T

2.3.4 Energy Method

We again investigate the initial/boundary value problem $\begin{cases} u_t - \Delta u = f & in \quad U_T \\ u = g & on \quad \Gamma_T \end{cases}$ and prove alternatively by integration by parts.

Theorem 2.27 (Uniqueness with Energy Method). *There exist only one solution $u \in C_1^2(\overline{U}_T)$ of initial/boundary-value problem.*

Proof. If $\widetilde{u}$ is another solution to the problem, we construct $w := u - \widetilde{u}$ as solution. Define energy $e(t) := \int_U w^2(x,t)dx$ for $0 \leq t \leq T$, then

$$\frac{de}{dt} = 2\int_U w w_t dx \tag{10}$$

$$= 2\int_U w \Delta w dx \tag{11}$$

$$= -2\int_U |Dw|^2 dx \leq 0 \tag{12}$$

Thus $e(t) \leq e(0) = 0$ for $0 \leq t \leq T$. □

A more subtle question would be about uniqueness backward in time for heat equation. In particular, we do not assume solutions coincide at time $t = 0$.

Theorem 2.28 (Backward Uniqueness)**.** *Let $u, \widetilde{u} \in C^2(U_T)$ respectively solve* $\begin{cases} u_t - \Delta u = 0 & in \quad U_T \\ u = g & on \quad \partial U \times [0,T] \end{cases}$ *and* $\begin{cases} \widetilde{u}_t - \Delta \widetilde{u} = 0 & in \quad U_T \\ \widetilde{u} = g & on \quad \partial U \times [0,T] \end{cases}$*, if we further have $u(x,T) = \widetilde{u}(x,T)$ for $x \in U$, then we obtain $u \equiv u$ within U_T.*

In other words, if two temperature distributions on U agree at some time $T > 0$ and have had the same boundary values for times $0 \le t \le T$, then these temperatures must have been identically equal within U at all earlier times.

2.4 Wave Equation

We study wave equation $u_{tt} - \Delta u = 0$ and the nonhomogeneous wave equation $u_{tt} - \Delta u = f$ subject to initial and boundary conditions.

2.4.1 Solution by Spherical Means

Solution for n = 1, d'Alembert's formula We first solve for $\begin{cases} u_{tt} - u_{xx} = 0 & in \quad \mathbb{R} \times (0,\infty) \\ u = g, u_t = h & on \quad \mathbb{R} \times \{t = 0\} \end{cases}$ We define $v(x,t) := (\frac{\partial}{\partial t} - \frac{\partial}{\partial x})u(x,t)$ and solve for transport equation. Our final result turns out to be

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy \qquad (x \in \mathbb{R}, t \ge 0)$$

We further solve the initial/boundary-value problem on half-line $\mathbb{R}_+ = \{x > 0\}$ $\begin{cases} u_{tt} - u_{xx} = 0 & in \quad \mathbb{R}_+ \times (0,\infty) \\ u = g, u_t = h & on \quad \mathbb{R}_+ \times \{t = 0\} \\ u = 0 & on \quad \{x = 0\} \times (0,\infty) \end{cases}$
We apply odd extension to u, g, h and convert our problem into solving extended solutions on $\mathbb{R}$, then apply d'Alembert's formula and arrive at

$$u(x,t) = \begin{cases} \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy & if \quad 0 \le t \le x \\ \frac{1}{2}[g(x+t) - g(t-x)] + \frac{1}{2}\int_{-x+t}^{x+t} h(y)dy & if \quad 0 \le x \le t \end{cases} \tag{13}$$

Now we need tools for solving higher dimension wave equations with $n \ge 2, m \ge 2$ and $u \in C^m(\mathbb{R}^n \times [0,\infty))$ solving $\begin{cases} u_{tt} - \Delta u = 0 & in \quad \mathbb{R}^n \times (0,\infty) \\ u = g, u_t = h & on \quad \mathbb{R}^n \times \{t = 0\} \end{cases}$ We first study average of u over certain spheres taken as functions of time t and radius r. Our idea lies in applying Euler-Poisson-Darboux equation to convert odd n into ordinary one-dimension wave equations. We begin with useful notations

Definition 2.29. *Let $x \in \mathbb{R}^n$, $t > 0$, $r > 0$, define $U(x;r,t) := \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(x,r)} u(y,t)dS(y)$ as average of u over sphere $\partial B(x,r)$. Similarily, define $G(x;r) = \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(x,r)} g(y)dS(y)$, $H(x;r) = \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(x,r)} h(y)dS(y)$*

We discover the Euler-Poisson-Darboux equation in spherical means

Theorem 2.30 (Euler-Poisson-Darboux equation)**.** *Fix $x \in \mathbb{R}^n$, and let u solve* $\begin{cases} u_{tt} - \Delta u = 0 & in \quad \mathbb{R}^n \times (0,\infty) \\ u = g, u_t = h & on \quad \mathbb{R}^n \times \{t = 0\} \end{cases}$*, then $U \in C^m(\overline{\mathbb{R}}_+ \times [0,\infty))$ and solve* $\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r}U_r = 0 & in \quad \mathbb{R}_+ \times (0,\infty) \\ U = G, U_t = H & on \quad \mathbb{R}_+ \times \{t = 0\} \end{cases}$

Solution for n = 3, Kirchhoff's formula Let $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ solve $\begin{cases} u_{tt} - \Delta u = 0 & in \quad \mathbb{R}^3 \times (0,\infty) \\ u = g, u_t = h & on \quad \mathbb{R}^3 \times \{t = 0\} \end{cases}$ then we set $\widetilde{U} := rU, \widetilde{G} := rG, \widetilde{H} := rH$. We notice that $\widetilde{U}$ in fact solves $\begin{cases} \widetilde{U}_{tt} - \widetilde{U}_{xx} = 0 & in \quad \mathbb{R}_+ \times (0,\infty) \\ \widetilde{U} = \widetilde{G}, \widetilde{U}_t = \widetilde{H} & on \quad \mathbb{R}_+ \times \{t = 0\} \\ \widetilde{U} = 0 & on \quad \{r = 0\} \times (0,\infty) \end{cases}$

We directly apply solution to reflection method for $0 \leq r \leq t$, i.e., $\widetilde{U}(x;r,t) = \frac{1}{2}[\widetilde{G}(r+t) - \widetilde{G}(t-r)] + \frac{1}{2}\int_{-r+t}^{r+t} \widetilde{H}(y)dy$ and observe a relation $u(x,t) = \lim_{r\to 0^+} \frac{\widetilde{U}(x;r,t)}{r} = \widetilde{G}'(t) + \widetilde{H}(t)$. We finally obtain Kirchhoff's formula for solution of initial-value problem in $n = 3$

$$u(x,t) = \frac{1}{3\alpha(3)t^2}\int_{\partial B(x,t)} th(y) + g(y) + Dg(y)\cdot(y-x)dS(y) \qquad (x\in\mathbb{R}^3, t>0)$$

Solution for n = 2, Poisson's formula Let $u \in C^2(\mathbb{R}^2\times[0,\infty))$ solve $\begin{cases} u_{tt} - \Delta u = 0 & in \quad \mathbb{R}^2\times(0,\infty) \\ u = g, u_t = h & on \quad \mathbb{R}^2\times\{t=0\}\end{cases}$ we solve the problem by regarding it as problem for $n = 3$ with third spatial variable x_3 not appearing. We write $\overline{u}(x_1,x_2,x_3,t) := u(x_1,x_2,t)$, $\overline{g}(x_1,x_2,x_3) := g(x_1,x_2)$, $\overline{h}(x_1,x_2,x_3) := h(x_1,x_2)$ and directly apply Kirchhoff's formula. We obtain $u(x,t) = \frac{\partial}{\partial t}(\frac{t}{4\pi t^2}\int_{\partial\overline{B}(\overline{x},t)}\overline{g}d\overline{S}) + \frac{t}{4\pi t^2}\int_{\partial\overline{B}(\overline{x},t)}\overline{h}d\overline{S}$. It suffices to simplify the above expressions by parametrization the surface measures. We eventually arrive at Poisson's formula

$$u(x,t) = \frac{1}{2\alpha(2)t^2}\int_{B(x,t)} \frac{tg(y) + t^2h(y) + tDg(y)\cdot(y-x)}{(t^2 - |y-x|^2)^{\frac{1}{2}}}dy \qquad (x\in\mathbb{R}^2, t>0)$$

The idea of solving the problem for $n = 3$ and then dropping to $n = 2$ is known as the method of descent.
Solution for odd n We apply similar idea by converting the problem to spherical means with Euler-Poisson-Darboux equation. The difference would be using more complicated identities for defining $\widetilde{U}$. We give the explicit expressions as $\begin{cases} \widetilde{U}(r,t) := (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}U(x;r,t)) \\ \widetilde{G}(r) := (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}G(x;r)) \\ \widetilde{H}(r) := (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}H(x;r))\end{cases}$ for $r > 0, t \geq 0$.

We check they satisfy $\begin{cases} \widetilde{U}_{tt} - \widetilde{U}_{xx} = 0 & in \quad \mathbb{R}_+\times(0,\infty) \\ \widetilde{U} = \widetilde{G}, \widetilde{U}_t = \widetilde{H} & on \quad \mathbb{R}_+\times\{t=0\} \\ \widetilde{U} = 0 & on \quad \{r=0\}\times(0,\infty)\end{cases}$ and so by similar method of solving for reflection method with $n = 1$, we arrive at representation formula for odd dimensions

$$u(x,t) = \frac{1}{\gamma_n}[(\frac{\partial}{\partial t})(\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-3}{2}}(\frac{1}{n\alpha(n)t}\int_{\partial B(x,t)} gdS) + (\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-3}{2}}(\frac{1}{n\alpha(n)t}\int_{\partial B(x,t)} hdS)]$$

$(n \text{ is odd and } \gamma_n = (n-2)\cdots 3\cdot 1)$

for $x \in \mathbb{R}^n, t > 0$. Notice that in order to compute $u(x,t)$, we only need information on g, h and their derivatives on $\partial B(x,t)$, but not the entire ball $B(x,r)$. Also, for $n > 1$, a solution of wave equation may not be as smooth as its initial value g.
Solution for even n We again apply method of descent, by defining $\overline{u}(x_1,\cdots,x_{n+1},t) := u(x_1,\cdots,x_nt)$, $\overline{g}(x_1,\cdots,x_{n+1}) := g(x_1,\cdots,x_n)$, $\overline{h}(x_1,\cdots,x_{n+1}) := h(x_1,\cdots,x_n)$. We plug in solution for odd dimensions as obtain $u(x,t) = \frac{1}{\gamma_{n+1}}[(\frac{\partial}{\partial t})(\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-2}{2}}(\frac{t^{n-1}}{(n+1)\alpha(n+1)t^n}\int_{\partial\overline{B}(x,t)}\overline{g}d\overline{S}) + (\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-2}{2}}(\frac{t^{n-1}}{(n+1)\alpha(n+1)t^n}\int_{\partial\overline{B}(x,t)}\overline{h}d\overline{S})]$. We again simplify by parametrization, and so we obtain the representation formula for even dimensions

$$u(x,t) = \frac{1}{\gamma_n}[\frac{\partial}{\partial t}(\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-2}{2}}(\frac{1}{\alpha(n)}\int_{B(x,t)}\frac{g(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy) + (\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-2}{2}}(\frac{1}{\alpha(n)}\int_{B(x,t)}\frac{h(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy)]$$

$(n \text{ is even and } \gamma_n = n\cdot(n-2)\cdots 4\cdot 2)$

for $x \in \mathbb{R}^n, t > 0$.
We notice that if n is odd and $n \geq 3$, data g and h at given point $x \in \mathbb{R}^n$ affect the solution u only on the boundary $\{(y,t)|t>0, |x-y| = t\}$ of the cone $C = \{(y,t)|t > 0, |x-y| < t\}$. On the other hand, if n is even, then data g and h affect u within all of C. Intuitively, a disturbance originating at x propagates along a sharp wavefront in odd dimensions, but in even dimensions it continues to have effect even after the leading edge of the wavefront passes. This is called **Huygens' Principle**.

2.4.2 Nonhomogeneous problem

We study the **initial-value problem for nonhomogeneous equation** $\begin{cases} u_{tt} - \Delta u = f & in \quad \mathbb{R}^n\times(0,\infty) \\ u = 0, u_t = 0 & on \quad \mathbb{R}^n\times\{t=0\}\end{cases}$
Again, we construct the solution by using $Duhamel's principle$. We define $u(x,t;s)$ as solution to homogenous problem $\begin{cases} u_{tt}(;s) - \Delta u(;s) = 0 & in \quad \mathbb{R}^n\times(0,\infty) \\ u(;s) = 0, u_t(;s) = f(,s) & on \quad \mathbb{R}^n\times\{t=s\}\end{cases}$ and set $u(x,t) := \int_0^t u(x,t;s)ds$ for $x \in \mathbb{R}^n, t \geq 0$. We

verify that it is indeed solution.

For **general nonhomogeneous problem**, we construct u as sum of solution to $\begin{cases} u_{tt} - \Delta u = 0 & in \quad \mathbb{R}^n \times (0, \infty) \\ u = g, u_t = h & on \quad \mathbb{R}^n \times \{t = 0\} \end{cases}$ and solution to $\begin{cases} u_{tt} - \Delta u = f & in \quad \mathbb{R}^n \times (0, \infty) \\ u = 0, u_t = 0 & on \quad \mathbb{R}^n \times \{t = 0\} \end{cases}$.

2.4.3 Energy Method

We find that wave equation is nicely behaved for all n w.r.t. certain integral energy norms. We first let $U \subset \mathbb{R}^n$ be bounded, open set with smooth boundary ∂U, and set $U_T = U \times (0, T]$, $\Gamma_T = \overline{U}_T - U_T$ for $T > 0$. We wish to study uniqueness to initial/boundary-value problem $\begin{cases} u_{tt} - \Delta u = f & in \quad U_T \\ u = g & on \quad \Gamma_T \\ u_t = h & on \quad U \times \{t = 0\} \end{cases}$

Theorem 2.31 (Uniqueness for Wave Equation)**.** *There exists at most one solution $u \in C^2(\overline{U}_T)$ to the initial/boundary-value problem.*

Proof. Let $\widetilde{u} be another solution, so w :=$ u - $\widetilde{u}$ solves homogeneous initial/boundary-value problem. We define our energy $E(t) := \frac{1}{2} \int_U w_t^2(x, t) + |Dw(x, t)|^2 dx$ for $0 \leq t \leq T$. Then we compute

$$\frac{d}{dt} E(t) = \int_U w_t w_{tt} + Dw \cdot Dw_t dx \tag{14}$$

$$= \int_U w_t(w_{tt} - \Delta w) dx = 0 \tag{15}$$

So $E(t) = E(0) = 0$ for any $0 \leq t \leq T$, thus $w = u - \widetilde{u} \equiv 0$ in U_T. □

Another illustration of energy method would be examining the domain of dependence for solutions to wave equation. We let $u \in C^2$ solve $u_{tt} - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$. Fix $_0 \in \mathbb{R}^n, t_0 > 0$, we define the backward wave cone with apex(x_0, t_0) as $K(x_0, t_0) := \{(x, t) | 0 \leq t \leq t_0, |x - x_0| \leq t_0 - t\}$.

Theorem 2.32 (Finite Propagation Speed)**.** *If $u \equiv u_t \equiv 0$ on $B(x_0, t_0) \times \{t = 0\}$, then $u \equiv 0$ within the cone $K(x_0, t_0)$.*

The proof is essentially done by defining local energy $e(t) := \frac{1}{2} \int_{B(x_0, t_0 - t)} u_t^2(x, t) + |Du(x, t)|^2 dx$ for $0 \leq t \leq t_0$ and computing $\frac{d}{dt} e(t)$ to observe $e(t) \leq e(0) = 0$ for all $0 \leq t \leq t_0$. We see that any disturbance originating outside $B(x_0, t_0)$ has no effect on the solution within the cone $K(x_0, t_0)$, thus the solution has finite propagation speed.

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4 References

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