Nonuniqueness of Mild Solutions to incompressible 3D Navier-Stokes Equations.

incompressible NS. $\begin{cases} \partial_t v + \operatorname{div}(v\otimes v) + \nabla p - \nu\Delta v = 0 \\ \operatorname{div} v = 0. \end{cases}$

<u>Def</u> $v \in C_t^0 L_x^2$ is mild solution to NS (on $[0,1]\times\mathbb{T}^3$)
if $\forall\, t\in[0,1]$.

- $v(\cdot,t)$ weakly divergence free.
  $\forall\, \phi\in C_0^\infty(\mathbb{T}^3)$ scalar. $\int_{\mathbb{T}^3} v\cdot\nabla\phi = 0.$
- $v(\cdot,t)$ zero mean. $\fint_{\mathbb{T}^3} v = 0.$
- $v(\cdot,t)$ solves NS in $D'([0,1]\times\mathbb{T}^3)$.
  $\forall\, \varphi\in C_0^\infty([0,1]\times\mathbb{T}^3;\mathbb{R}^3)$. s.t. $\operatorname{div}\varphi = 0$ $\forall\, t$.
  we have $\int_0^1\int_{\mathbb{T}^3} v\cdot(\partial_t\varphi + (v\cdot\nabla)\varphi + \nu\Delta\varphi)\,dx\,dt = 0.$

<u>THM</u>. $\exists\, \beta>0$ small and $\nu\in(0,1]$ small s.t. $\exists\, v$ mild solution to NS
in $C^0([0,1];H^\beta(\mathbb{T}^3)) \cap C^0([0,1];W^{1,1+\beta}(\mathbb{T}^3))$
s.t. $\|v(\cdot,1)\|_{L^2} > 2\|v(\cdot,0)\|_{L^2}.$

- notice $v\equiv 0$ is automatically mild solution.
  Hence we have non-uniqueness.
- also such mild solution isn't equal to any Leray-Hopf weak solution arising from $v|_{t=0} \in L_0^2$
  due to violation of strong energy inequality
  $\left(\frac{1}{2}\int_{\mathbb{T}^3} |v(x,1)|^2 dx + \int_0^1\int_{\mathbb{T}^3} \nu|\nabla v(x,s)|^2 dx\,ds \le \frac{1}{2}\int_{\mathbb{T}^3}|v(x,0)|^2 dx. \right)$
  where $|\nabla v|^2 = \partial_j v^i \partial_j v^i$.

How to Prove? Intermittent Convex Integration.

Prop if $\exists$ $(v_q, \mathring{R}_q)$ solving NS-Reynolds System.

$$\begin{cases} \partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q - \nu \Delta v_q = \operatorname{div} \mathring{R}_q \\ \operatorname{div} v_q = 0. \end{cases} \quad (1)$$

s.t
$$\begin{cases} \| v_q \|_{L^2} \le 1 - \delta_q^{1/2} \\ \| v_q \|_{C^1_{t,x}} \le \lambda_q^4 \\ \| \mathring{R}_q \|_{L^1} \le_{CR} \delta_{q+1} \end{cases}$$

then $\exists$ $(v_{q+1}, \mathring{R}_{q+1})$ solving NS-Reynolds (incompressible) and above estimates at $q+1$.

st. $\| v_{q+1} - v_q \|_{L^2} \le \delta_{q+1}^{1/2}$

Here $\lambda_q = 2\pi\, a^{(b^q)}$, $\delta_q = \lambda_q^{-2\beta}$.

- How to define v_{q+1}? $v_{q+1} = v_\ell + w_{q+1}$.

$\to$ so $v_{q+1} - v_q = v_\ell - v_q + w_{q+1}$ to estimate L_2 norm.

$\to$ v_ℓ solves

$$\begin{cases} \partial_t v_\ell + \operatorname{div}(v_\ell \otimes v_\ell) + \nabla p_\ell - \nu \Delta v_\ell = \operatorname{div}(\mathring{R}_\ell + R_{com}) \\ \operatorname{div} v_\ell = 0 \end{cases} \quad (2)$$

for $v_\ell = (v_q *_x \phi_\ell) *_t \varphi_\ell$. $\mathring{R}_\ell = (\mathring{R}_q *_x \phi_\ell) *_t \varphi_\ell$

(because $\partial_t \nabla ; \Delta$ commutes with convolution.

and. $\operatorname{div}(v_\ell \otimes v_\ell) = \operatorname{div}(v_\ell \mathring{\otimes} v_\ell)$.)

where $R_{com} = v_\ell \mathring{\otimes} v_\ell - ((v_q \mathring{\otimes} v_q) *_x \phi_\ell) *_t \varphi_\ell$

So let (1) − (2). where (1) at $q+1$ level.

$$\operatorname{div} \mathring{R}_{q+1} - \nabla p_{q+1} = \partial_t w_{q+1} + \operatorname{div}(v_\ell \otimes w_{q+1} + w_{q+1} \otimes v_\ell + w_{q+1} \otimes w_{q+1}) - \underline{\nu \Delta w_{q+1}} - \nabla p_\ell + \operatorname{div}(\mathring{R}_\ell + R_{com}) \quad (3)$$

different from Euler.

1) Our target is to estimate $\mathring{R}_{q+1}$ in L^1 base.
let's look at how $\mathcal{U}\Delta w_{q+1}$ behaves.
first apply $\mathcal{R}$ inverse elliptic operator

$$\|\mathcal{R}\Delta w_{q+1}\|_{L^p} \sim \|\nabla w_{q+1}\|_{L^p} \sim \|w_{q+1}\|_{W^{1,p}}$$

for w_{q+1}, its principal part $w_{q+1}^{(p)}$ dominates.

and similarly (as Euler's) defined as

$$w_{q+1}^{(p)} = \sum_{\xi\in\Lambda} a_{(\xi)} W_{(\xi)}$$

so $\sim \sum \|a_{(\xi)} W_{(\xi)}\|_{W^{1,p}} \lesssim \|a\|_{C_{t,x}} \|W_{(\xi)}\|_{W^{1,p}}$

first order derivative gives λ_{q+1} (big?)
if we're still using stationary Mikado flows
then $W_{(\xi)} = \xi\,\phi_{(\xi)}$ in L^p has no gain.
so $\|W_{(\xi)}\|_{W^{1,p}}$ remains big.
$\Rightarrow$ think of new method of defining building blocks
make use of smaller frequencies concentration

$$\lambda_{q+1}^{-1} \ll r_\perp \ll r_\parallel \ll 1$$

$$\phi_{(\xi)}(x) = \phi(n_*\lambda(x-\alpha_\xi)\cdot A_\xi,\ n_*\lambda(x-\alpha_\xi)\cdot(\xi\times A_\xi))$$

$$\downarrow$$

- $\phi_{(\xi)}(x) = \phi_{r_\perp}(n_* r_\perp \lambda(x-\alpha_\xi)\cdot A_\xi,\ n_* r_\perp \lambda(x-\alpha_\xi)\cdot(\xi\times A_\xi))$

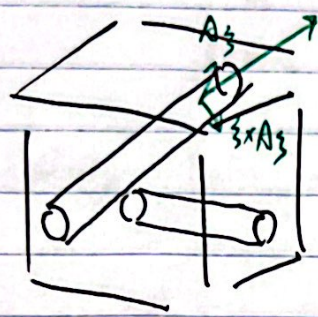
  where $\phi_{r_\perp} = \frac{1}{r_\perp}\phi\left(\frac{x_1}{r_\perp}, \frac{x_2}{r_\perp}\right)$
- $\psi_{(\xi)}(x,t) = \psi_{r_\parallel}(n_* r_\perp \lambda(x\cdot\xi + \mu t))$

  where $\psi_{r_\parallel} = \frac{1}{r_\parallel^{1/2}}\psi\left(\frac{x_3}{r_\parallel}\right)$

Define $W_{(\xi)} = \xi\,\phi_{(\xi)}\psi_{(\xi)}$ intermittent jets.

let's make sense of intermittent jets.

old Mikado flows. $W_{(\xi)} = \xi \phi_{(\xi)} = \xi \phi(n_* \lambda(x-\alpha_\xi)\cdot A_\xi, n_*\lambda(x-\alpha_\xi)\cdot \xi\times A_\xi)$

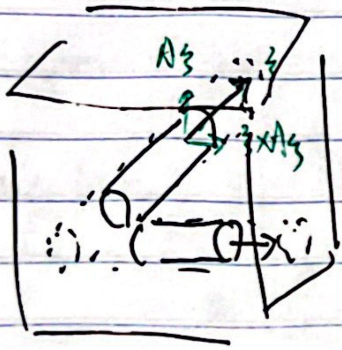


$(\pi/\lambda)^3$ box.

- diameter of tubes $\sim \lambda^{-1}$.
- n_* smallest integer s.t. $\{n_*\xi, n_* A_\xi, n_*\xi\times A_\xi\} \in \mathbb{Z}^3$.
- α_ξ shifts to be disjoint.

intermittent jets. $W_{(\xi)} = \xi \phi_{(\xi)} \psi_{(\xi)}$

$$= \xi \cdot \frac{1}{r_\perp} \phi(n_* \underline{\lambda}(x-\alpha_\xi)\cdot A_\xi, n_*\lambda(x-\alpha_\xi)\cdot \xi\times A_\xi)\cdot \frac{1}{r_\parallel^{1/2}} \psi\Big(n_* \underbrace{\frac{r_\perp}{r_\parallel}}_{\text{intermittency}} \lambda(x\cdot\xi + \mu t)\Big)$$



$(\pi/\lambda r_\perp)^3$ box

- diameter of tubes $\sim \lambda^{-1}$
- length of tube $\sim \frac{r_\parallel}{r_\perp}\lambda^{-1}$
- μ - time oscillation parameter

$\lambda r_\perp \in \mathbb{N}$ manually designed.
smaller boxes.

support of $W_{(\xi)} \sim (\lambda r_\perp)^3 \cdot (\lambda^{-1})^2 \cdot \frac{r_\parallel}{r_\perp}\lambda^{-1}$

$= r_\parallel r_\perp^2$ measure w.r.t. $\mathbb{T}^3$.

→ compute
$$\begin{cases} \|\nabla^N \partial_t^M \psi_{(\xi)}\|_{L^p} \lesssim r_\parallel^{1/p-1/2} \left(\frac{r_\perp\lambda}{r_\parallel}\right)^N \left(\frac{r_\perp}{r_\parallel}\lambda\mu\right)^M. \\ \|\nabla^N \phi_{(\xi)}\|_{L^p} \lesssim r_\perp^{2/p-1} \lambda^N \end{cases}$$

- $\|\psi_{(\xi)}\|_{L^p} = \left(\int \psi_{r_\parallel}^p\right)^{1/p} \lesssim \frac{1}{r_\parallel^{1/2}}\left(\int \psi^p \underline{r_\parallel}\right)^{1/p}$ Jacobian

b/c $\psi_{r_\parallel} = \frac{1}{r_\parallel^{1/2}}\psi\left(\frac{x_3}{r_\parallel}\right)$

- $\|\phi_{(\xi)}\|_{L^p} \lesssim \frac{1}{r_\perp}\left(\int \phi^p r_\perp^2\right)^{1/p}$ $\phi_{r_\perp} = \frac{1}{r_\perp}\phi\left(\frac{x_1}{r_\perp}, \frac{x_2}{r_\perp}\right)$

$$\Rightarrow \|\nabla^N \partial_t^M W_{(\xi)}\| \lesssim r_\perp^{2/p-1} r_\parallel^{1/p-1/2} \lambda^N \left(\frac{r_\perp}{r_\parallel}\lambda\mu\right)^M. \quad (4)$$

Hence for $p=\infty$ as before, $r_\perp^{-1} r_{\parallel}^{-1/2}$ as positive power of λ is by no means small.

However for $p \in [1,2)$, $r_\perp^{\frac{2}{p}-1} r_\parallel^{\frac{1}{p}-\frac{1}{2}}$ gives negative power of λ in which we can manipulate.

Hence we do Convex integration in L^1 base for $\mathring{R}_q$

- The above deals with error $\nabla \cdot w_{q+1}$.

Oscillation ERROR.

2) Recall to deal with mollified stress $\mathring{R}_\ell$ we need low frequency part for $w_{q+1}^{(p)} \otimes w_{q+1}^{(p)}$.

upon defining $w_{q+1}^{(p)} = \sum_{\xi \in \Lambda} a_{(\xi)} W_{(\xi)}$.

we have $w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} = \sum_{\xi \in \Lambda} a_{(\xi)}^2 W_{(\xi)} \otimes W_{(\xi)}$

$$= \sum_{\xi \in \Lambda} a_{(\xi)}^2 \underbrace{\fint_{\mathbb{T}^3} W_{(\xi)} \otimes W_{(\xi)}}_{\text{lower mode}} + \sum_{\xi \in \Lambda} a_{(\xi)}^2 \underbrace{\mathbb{P}_{\neq 0} W_{(\xi)} \otimes W_{(\xi)}}_{\text{higher mode}}.$$

Q How do we define amplitude $a_{(\xi)}$ s.t. it cancels $\mathring{R}_\ell$? (via lower frequency).

- first notice $\fint_{\mathbb{T}^3} W_{(\xi)} \otimes W_{(\xi)} = \xi \otimes \xi \fint_{\mathbb{T}^3} \phi^2 \psi^2 = \xi \otimes \xi$.
- We have linear algebra lemma.

$\exists \Lambda \subseteq \mathbb{S}^2 \cap \mathbb{Q}^3$ s.t. $\forall \xi \in \Lambda$. $\exists \gamma_\xi \in C^\infty : B_{1/2}(\mathrm{Id}) \to \mathbb{R}$

s.t. $R = \sum_{\xi \in \Lambda} \gamma_\xi^2 (R)\, \xi \otimes \xi \quad \forall R \in \overline{B_{1/2}(\mathrm{Id})}$

we think about
- How to define $a_{(\xi)}$ using γ_ξ
- What's our choice of R so that $R \in \overline{B_{1/2}(\mathrm{Id})}$

Recall in our purpose

$$\|v_{q+1}-v_q\|_{L^2} \le \|w_{q+1}\|_{L^2} + \|v_\ell - v_q\|_{L^2} \le \delta_{q+1}^{1/2}$$

we want $\|w_{q+1}\|_{L^2} \le \delta_{q+1}^{1/2}$

$$\sim \|w_{q+1}^{\varphi_1}\|_{L^2} \lesssim \delta_{q+1}^{1/2}$$

$\|$ $\quad \delta_{q+1}^{1/2}$ has to come from $a_{(\xi)}$

$$\sum_{\xi\in\Lambda} \| a_{(\xi)} W_{(\xi)} \|_{L^2}.$$

<u>lemma</u> L^p De-correlation.

$a_{(\xi)} - \mathbb{T}^3$. $\quad W_{(\xi)} - (\mathbb{T}/\lambda r_\perp)^3$.

freq $a_{(\xi)}$ << freq $W_{(\xi)}$ $\lambda r_\perp$.

$$\Rightarrow \quad \|a_{(\xi)} W_{(\xi)}\|_{L^2} \le \|a_{(\xi)}\|_{L^2} \|W_{(\xi)}\|_{L^2}.$$

$$\boxed{\lesssim} \|\mathring{R}_q\|_{L^1}^{1/2} \lesssim C_R^{1/2} \delta_{q+1}^{1/2}.$$

roughly $\int a_{(\xi)}^2 \le \int \mathring{R}_q \lesssim C_R \delta_{q+1}$

So Design $a_{(\xi)} = \rho^{1/2} \gamma_\xi \left(Id - \frac{\mathring{R}_\ell}{\rho} \right)$, to cancel $\rho^{1/2}$.

to cancel $\mathring{R}_\ell$.

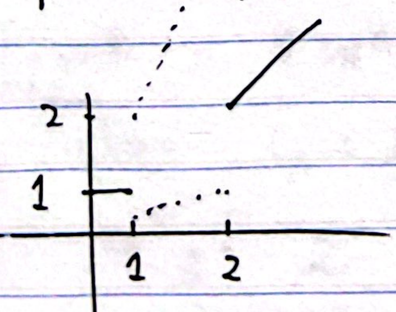
where $\rho \sim C_R \delta_{q+1}$.

How to ensure $\left| \frac{\mathring{R}_\ell}{\rho} \right| < \frac{1}{2}$?

let $\rho = 4 C_R \delta_{q+1} \chi\left(\frac{|\mathring{R}_\ell|}{C_R \delta_{q+1}} \right)$ where $\chi = \begin{cases} 1 & 0 \le z \le 1 \\ z & z \ge 2 \end{cases}$

and $\chi \in (\frac{z}{2}, 2z)$ if $z \in (1,2)$

smooth



Hence compute.

$$\left|\frac{\mathring{R}_\ell}{\rho}\right| = \frac{|\mathring{R}_\ell|}{4C_R\delta_{q+1}\chi\left(\frac{|\mathring{R}_\ell|}{C_R\delta_{q+1}}\right)} = \frac{1}{4}\cdot\frac{\frac{|\mathring{R}_\ell|}{C_R\delta_{q+1}}}{\chi\left(\frac{|\mathring{R}_\ell|}{C_R\delta_{q+1}}\right)}.$$

notice χ has lower bound.

- if $\frac{|\mathring{R}_\ell|}{C_R\delta_{q+1}} \le 1$. $\chi = 1$. so $\left|\frac{\mathring{R}_\ell}{\rho}\right| \le \frac{1}{4} < \frac{1}{2}$.
- if $\frac{|\mathring{R}_\ell|}{C_R\delta_{q+1}} \in (1,2)$ $\chi \ge \frac{|\mathring{R}_\ell|}{C_R\delta_{q+1}}\cdot\frac{1}{2}$. so $\left|\frac{\mathring{R}_\ell}{\rho}\right| < \frac{1}{2}$.
- if $\frac{|\mathring{R}_\ell|}{C_R\delta_{q+1}} \ge 2$. $\chi = \frac{|\mathring{R}_\ell|}{C_R\delta_{q+1}}$ so $\left|\frac{\mathring{R}_\ell}{\rho}\right| = \frac{1}{4} < \frac{1}{2}$

→ Meanwhile let's make sense of C_R universal constant.

$$\|a_{(\xi)}\|_{L^2} = \left(\int_{\mathbb{T}^3} \rho\,\gamma_\xi^2\right)^{1/2} \le \|\gamma_\xi\|_{C^0}\left(\int\rho\right)^{1/2}.$$

- from linear algebra $\|\gamma_\xi\|_{C^0} \le \frac{M}{C_\Lambda}$

while - $\|\rho\|_{L^p} = 4C_R\delta_{q+1}\left(\int_{\mathbb{T}^3}|\chi|^p\left(\frac{|\mathring{R}_\ell|}{C_R\delta_{q+1}}\right)dx\right)^{1/p}$.

$$\overset{\text{Minkowski}}{\le} \cdots\left(\left(\int_{\mathbb{T}^3}\mathbb{1}_{\left\{\frac{|\mathring{R}_\ell|}{C_R\delta_{q+1}}\le 1\right\}}\right)^{1/p} + \left(\int_{\mathbb{T}^3}\mathbb{1}_{\left\{\frac{|\mathring{R}_\ell|}{C_R\delta_{q+1}}> 1\right\}}|\chi|^p\right)^{1/p}\right)$$

$$\le 4C_R\delta_{q+1}\left((8\pi^3)^{\frac{1}{p}} + 2\frac{\|\mathring{R}_\ell\|_{L^p}}{C_R\delta_{q+1}}\right)$$

$$\le 16\left(C_R\delta_{q+1}(8\pi^3)^{\frac{1}{p}} + \|\mathring{R}_\ell\|_{L^p}\right)$$

take $p=1$ and $\frac{1}{2}$ outside, note $\|\mathring{R}_\ell\|_{L^1} \lesssim C_R\delta_{q+1}$

$$\left(\int\rho\right)^{1/2} \lesssim 4C_R^{1/2}\delta_{q+1}^{1/2}\,\underline{(8\pi^3+1)^{1/2}}.$$

choose $C_\Lambda = 8|\Lambda|(1+8\pi^3)^{1/2}$

so $\|a_{(\xi)}\|_{L^2} \lesssim \frac{1}{2|\Lambda|}C_R^{1/2}\delta_{q+1}^{1/2}M$. → take C_R small to absorb M.

$$\lesssim \frac{1}{2|\Lambda|}\delta_{q+1}^{1/2}$$

$|\Lambda|$ for $\sum_{\xi\in\Lambda}$

Now Return to low frequency cancellation.

$$w^{(p)}_{q+1} \otimes w^{(p)}_{q+1} + \mathring{R}_\ell$$

$$= \Big(\sum_{\xi\in\Lambda} a^2_{(\xi)} \fint_{\mathbb{T}^3} W_{(\xi)} \otimes W_{(\xi)} + \mathring{R}_\ell \Big) + \sum_{\xi\in\Lambda} a^2_{(\xi)} \mathbb{P}_{\neq 0} W_{(\xi)} \otimes W_{(\xi)}$$

$$\sum_{\xi\in\Lambda} \rho\, \gamma_\xi^2 (\mathrm{Id} - \tfrac{\mathring{R}_\ell}{\rho})\, \xi\otimes\xi$$

$$= \rho\,\mathrm{Id} - \mathring{R}_\ell .$$

$$= \rho\,\mathrm{Id} + \underbrace{\sum_{\xi\in\Lambda} a^2_{(\xi)} \mathbb{P}_{\neq 0} W_{(\xi)} \otimes W_{(\xi)}}_{\text{higher mode.}}$$

3) Closer Look Oscillation ERROR under "div"

$$\mathrm{div}\big(w^{(p)}_{q+1} \otimes w^{(p)}_{q+1} + \mathring{R}_\ell \big)$$

$$= \nabla \rho + \sum_{\xi\in\Lambda} \mathrm{div}\big(a^2_{(\xi)} \mathbb{P}_{\neq 0} W_{(\xi)} \otimes W_{(\xi)} \big)$$

$$= \nabla \rho + \sum_{\xi\in\Lambda} \mathbb{P}_{\neq 0} \nabla a^2_{(\xi)} \mathbb{P}_{\neq 0} W_{(\xi)} \otimes W_{(\xi)} + \sum_{\xi\in\Lambda} \mathbb{P}_{\neq 0}\, a^2_{(\xi)} \underline{\mathrm{div}\big(W_{(\xi)} \otimes W_{(\xi)}\big)}$$

(put $\mathbb{P}_{\neq 0}$ back b/c $\mathrm{div}(\mathring{R}) = \mathrm{div}(R)$)

Q How do we deal with $\mathrm{div}\big(W_{(\xi)} \otimes W_{(\xi)}\big)$?

Back when we used stationary Mikados.

$$\mathrm{div}\big(W_{(\xi)} \otimes W_{(\xi)}\big) = 0. \quad (\text{b/c } \nabla\cdot\phi = 0).$$

But for intermittent jets. $\mathrm{div}\big(W_{(\xi)} \otimes W_{(\xi)}\big) \neq 0.$

Let's calculate.

$$\operatorname{div}(W_{(\xi)} \otimes W_{(\xi)}) = \operatorname{div}(\phi^2_{(\xi)} \psi^2_{(\xi)} \, \xi \otimes \xi)$$

$$= \left(\partial_i(\phi^2_{(\xi)} \psi^2_{(\xi)}) \xi^j \xi^i \right)_{j=1,2,3}$$

$$= \left(\left(2\psi\psi_i \phi^2 + \cancel{2\phi\phi_i\psi^2}^{\,0}\right) \xi^j \xi^i \right)_{j=1,2,3}$$

$\nabla \cdot \phi = 0$

$$= 2\psi\phi^2 \left(\psi_i \xi^j \xi^i \right)_{j=1,2,3}$$

$$= 2\psi\phi^2 \left(\nabla\psi_{(\xi)} \cdot \xi \right) \xi$$

$$= 2\left(W_{(\xi)} \cdot \nabla\psi_{(\xi)} \right) \phi_{(\xi)} \xi \qquad (*)$$

now observe $\psi_{(\xi)} = \psi_{r_\parallel}(n_* r_\perp \lambda (x \cdot \xi + \mu t))$.

$$\Rightarrow \quad \frac{\partial}{\partial x_i} \psi_{(\xi)} = \psi'_{r_\parallel} \cdot n_* r_\perp \lambda \xi^i$$

$$\& \quad \frac{\partial}{\partial t} \psi_{(\xi)} = \cdots\cdots \mu.$$

hence since $|\xi|^2 = 1$.

$$\nabla\psi_{(\xi)} \cdot \xi = \frac{1}{\mu} \partial_t \psi.$$

$$\Rightarrow \quad (*) = 2\psi\phi^2 \frac{1}{\mu} \partial_t \psi \, \xi$$

$$= \frac{1}{\mu} \phi^2 \partial_t \psi^2 \, \xi = \underline{\frac{1}{\mu} \partial_t (\phi^2 \psi^2 \xi)}$$

Hence Rewrite $\sum_{\xi \in \Lambda} \mathbb{P}_{\neq 0} \, a^2_{(\xi)} \operatorname{div}(W_{(\xi)} \otimes W_{(\xi)})$

$$= \frac{1}{\mu} \sum_{\xi \in \Lambda} \mathbb{P}_{\neq 0} \left(a^2_{(\xi)} \partial_t (\phi^2_{(\xi)} \psi^2_{(\xi)} \xi) \right)$$

Recall if ∂_t hit on $\phi_{(\xi)}$ or $\psi_{(\xi)}$

we get positive powers of λ.

Hence we think about transferring the time derivative to $a_{(\xi)}$

By writing

$$\partial_t\left(a^2_{(\xi)}\phi^2_{(\xi)}\psi^2_{(\xi)}\xi\right) = \partial_t a^2_{(\xi)}\left(\phi^2\psi^2\xi\right) + a^2_{(\xi)}\partial_t\left(\phi^2\psi^2\xi\right)$$

make this pressure term — gain good estimate $\|a_{(\xi)}\|_{C^N}$.

<u>Solution</u> Define $\boxed{w^{(t)}_{q+1}} := -\frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}_H\mathbb{P}_{\neq 0}\left(a^2_{(\xi)}\phi^2_{(\xi)}\psi^2_{(\xi)}\xi\right)$

where $\mathbb{P}_H \equiv Id - \nabla\Delta^{-1}\mathrm{div}$ is Leray projector.

which commutes with ∂_t.

So

$$\partial_t w^{(t)}_{q+1} = -\frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}_H\mathbb{P}_{\neq 0}\,\partial_t\left(a^2_{(\xi)}\phi^2_{(\xi)}\psi^2_{(\xi)}\xi\right)$$

$$= \underline{(Id-\mathbb{P}_H)\frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}_{\neq 0}\,\partial_t\left(a^2_{(\xi)}\phi^2_{(\xi)}\psi^2_{(\xi)}\xi\right)}$$

$\hookrightarrow$ pressure term $=: \nabla P$.

$$- \frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}_{\neq 0}\,\partial_t\left(a^2_{(\xi)}\phi^2_{(\xi)}\psi^2_{(\xi)}\xi\right).$$

Hence

$$\partial_t w^{(t)}_{q+1} + \sum_{\xi\in\Lambda}\mathbb{P}_{\neq 0}\, a^2_{(\xi)}\,\mathrm{div}\left(W_{(\xi)}\otimes W_{(\xi)}\right)$$

$$= \nabla P - \frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}_{\neq 0}\left(\partial_t a^2_{(\xi)}\left(\phi^2_{(\xi)}\psi^2_{(\xi)}\xi\right)\right).$$

$\Rightarrow$ $\mathrm{div}\left(w^{(p)}_{q+1}\otimes w^{(p)}_{q+1} + \mathring{R}_\ell\right) + \partial_t w^{(t)}_{q+1}$

$$= \nabla p + \nabla P + \sum_{\xi\in\Lambda}\mathbb{P}_{\neq 0}\,\underline{\underline{\nabla a^2_{(\xi)}}}\,\mathbb{P}_{\neq 0}\, W_{(\xi)}\otimes W_{(\xi)} - \frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}_{\neq 0}\left(\underline{\underline{\partial_t a^2_{(\xi)}}}\left(\phi^2_{(\xi)}\psi^2_{(\xi)}\xi\right)\right)$$

remains to estimate.

Q. How to compute $\|a_{(\xi)}\|_{C^N_{t,x}}$

4) Recall $a_{(\xi)} = \rho(x,t)^{1/2}\gamma_\xi\left(Id - \frac{\mathring{R}_\ell}{\rho}\right)$

for $\rho = 4 c_R \delta_{q+1} \chi\left(\frac{|\mathring{R}_\ell|}{c_R\delta_{q+1}}\right)$

Lemma $\Phi : \Omega \to \mathbb{R}$, $u : \mathbb{R}^n \to \Omega$ smooth. $m \in \mathbb{N}\setminus\{0\}$.

$$[\Phi\circ u]_m \lesssim [\Phi]_1 [u]_m + \|D\Phi\|_{m-1}\|u\|_0^{m-1}[u]_m.$$

take $\Phi = c_R\delta_{q+1}\chi(\cdot)$ $u = \frac{1}{c_R\delta_{q+1}}|\mathring{R}_\ell|$

i) $[\rho]_N \lesssim \left(c_R\delta_{q+1} + c_R\delta_{q+1}\left(\frac{1}{c_R\delta_{q+1}}\|\mathring{R}_\ell\|_0\right)^{N-1}\right)\frac{1}{c_R\delta_{q+1}}[\mathring{R}_\ell]_N.$

- at $d=3$. $4 > \frac{3}{1}$. so $W^{4,1} \hookrightarrow C^0$.

  Hence $\|\mathring{R}_\ell\|_0 \lesssim \|\mathring{R}_\ell\|_{W^{4,1}} \lesssim \ell^{-4}\|\mathring{R}_q\|_{L^1}$

  $\leq \ell^{-4} c_R\delta_{q+1}$

- $[\mathring{R}_\ell]_N \lesssim \ell^{-N}[\mathring{R}_\ell]_0 \lesssim \ell^{-N}\ell^{-4} c_R\delta_{q+1}.$

So, $[\rho]_N \lesssim c_R\delta_{q+1}\left(1 + \ell^{-4N+4}\right)\ell^{-N}\ell^{-4}.$

$= c_R\delta_{q+1}\ell^{-N}\ell^{-4} + c_R\delta_{q+1}\ell^{-5N}.$

$\lesssim c_R\delta_{q+1}\ell^{-5N}\ell^{-4}$

ii) $\|a_{(\xi)}\|_{C^N} \lesssim \|\rho^{1/2}\|_{C^N}\|\gamma_\xi\|_{C^0} + \|\rho^{1/2}\|_{C^0}\|\gamma_\xi\|_{C^N}.$

$[\gamma_\xi]_0 \lesssim 1$

$[\rho^{1/2}]_{C^0} \leq 2\sqrt{c_R\delta_{q+1}}\,\chi^{1/2}\left(\frac{|\mathring{R}_\ell|}{c_R\delta_{q+1}}\right) \leq 2\sqrt{c_R\delta_{q+1}}\frac{\|\mathring{R}_\ell\|_0^{1/2}}{\sqrt{c_R\delta_{q+1}}} \lesssim \ell^{-2}(c_R\delta_{q+1})^{1/2}$

$[\rho^{1/2}]_{C^N} \leq \left([\sqrt{\rho}]_1[\rho]_N + \|\tfrac{1}{2}\tfrac{1}{\sqrt{\rho}}\|_{N-1}\|\rho\|_0^{N-1}[\rho]_N\right)$

here $[\sqrt{\rho}]_1 \lesssim \frac{1}{2}[\frac{1}{\sqrt{\rho}}]_0 \lesssim \frac{1}{\sqrt{c_R\delta_{q+1}}}$ (b/c $\chi \geq \frac{1}{2}$. so $\rho \geq 2c_R\delta_{q+1}$)

$\|\frac{1}{\sqrt{\rho}}\|_{N-1} \lesssim (c_R\delta_{q+1})^{-\frac{1}{2}-(N-1)} = (c_R\delta_{q+1})^{\frac{1}{2}-N}.$

while $\|\rho\|_0^{N-1} \lesssim \left(CR\delta_{q+1} \ell^{-4}\right)^{N-1} = (CR\delta_{q+1})^{N-1} \ell^{-4N+4}$.

$[\rho]_N \lesssim CR\delta_{q+1} \ell^{-5N} \ell^{-4}$.

Hence $[\rho^{1/2}]_{C^N} \lesssim \left((CR\delta_{q+1})^{-\frac{1}{2}} + (CR\delta_{q+1})^{\frac{1}{2}-N} (CR\delta_{q+1})^{N-1} \ell^{-4N+4}\right) CR\delta_{q+1} \ell^{-5N-4}$

$\lesssim (CR\delta_{q+1})^{1/2} \left(\ell^{-5N-4} + \ell^{-9N}\right)$

$\lesssim (CR\delta_{q+1})^{1/2} \ell^{-9N-4}$

$\Rightarrow \|a_{(\xi)}\|_{C^N} \lesssim CR^{1/2} \delta_{q+1}^{1/2} \ell^{-9N-4}$.

($[\gamma]_{C^N}$ is similar to $[\rho^{1/2}]_{C^N}$ and gets eaten)

5) Notice since $w_{q+1}^{(t)}$ includes $\mathbb{P}_H$.

$\operatorname{div} w_{q+1}^{(t)} = 0$.

But $\operatorname{div} w_{q+1}^{(p)}$ is by no means divergence-free

Hence $w_{q+1} := w_{q+1}^{(p)} + w_{q+1}^{(t)}$ not enough

to guarantee v_{q+1} divergence. free

we define. $w_{q+1}^{(c)}$ incompressibility corrector.

<u>Q</u> How $w_{q+1}^{(c)}$ defined ?

write $w_{q+1}^{(p)} + w_{q+1}^{(c)} = \sum_{\xi\in\Lambda} \operatorname{curl}\operatorname{curl}\left(a_{(\xi)} V_{(\xi)}\right)$ for some $V_{(\xi)}$

we check $\operatorname{curl}\operatorname{curl}\left(a_{(\xi)} V_{(\xi)}\right)$

$= \operatorname{curl}\left(a_{(\xi)} \operatorname{curl} V_{(\xi)} + \nabla a_{(\xi)} \times V\right)$.

$= \underline{a_{(\xi)} \operatorname{curl}\operatorname{curl} V_{(\xi)} +} \nabla a_{(\xi)} \times \operatorname{curl} V_{(\xi)}$

$+ \operatorname{curl}\left(\nabla a_{(\xi)} \times V\right)$

$a_{(\xi)} W_{(\xi)} + a_{(\xi)}\left(\operatorname{curl}\operatorname{curl} V_{(\xi)} - W_{(\xi)}\right)$

only part containing $a_{(\xi)}$.

$$\text{curl curl } V_{(\xi)} - W_{(\xi)}$$

Hence Define

$$\boxed{w_{q+1}^{(c)}} := \sum_{\xi\in\Lambda}\left[\operatorname{curl}(\nabla a_{(\xi)}\times V_{(\xi)}) + \nabla a_{(\xi)}\times \operatorname{curl} V_{(\xi)} + a_{(\xi)}\boxed{W^{(c)}_{(\xi)}}\right]$$

now what is $V_{(\xi)}$?

since both curl curl $V_{(\xi)}$ and $W_{(\xi)}$ appear in $w_{q+1}^{(c)}$

we set curl curl $V_{(\xi)} \sim \xi\psi_{(\xi)}\phi_{(\xi)}$.

let $\Phi_{(\xi)} = \Phi_{r_\perp}(n_* r_\perp\lambda(x-\alpha_\xi)\cdot A_\xi,\ n_* r_\perp\lambda(x-\alpha_\xi)\cdot(\xi\times A_\xi))$

where $\Phi_{r_\perp} = \frac{1}{r_\perp}\Phi\left(\frac{x_1}{r_\perp}, \frac{x_2}{r_\perp}\right)$

s.t. $\phi = -\Delta\Phi$. (so $\phi_{r_\perp} = -r_\perp^2\Delta\Phi_{r_\perp}$)

since two curls cost $(n_* r_\perp\lambda)^2$.

Define $\boxed{V_{(\xi)}} = \frac{1}{n_*^2\lambda^2}\Phi_{(\xi)}\psi_{(\xi)}\xi$

we calculate curl curl $V_{(\xi)} - W_{(\xi)} = W^{(c)}_{(\xi)}$.

$$= \frac{1}{n_*^2\lambda^2}\operatorname{curl}\left(\Phi_{(\xi)}\operatorname{curl}(\psi_{(\xi)}\xi) + \nabla\Phi_{(\xi)}\times(\psi_{(\xi)}\xi)\right) - W_{(\xi)}$$

here $\operatorname{curl}(\nabla\Phi_{(\xi)}\times(\psi_{(\xi)}\xi))$

$$= \underline{\nabla\Phi_{(\xi)}(\nabla\cdot(\psi_{(\xi)}\xi))} - \psi_{(\xi)}\xi(\nabla\cdot\nabla\Phi_{(\xi)}) + (\psi_{(\xi)}\xi\cdot\nabla)(\nabla\Phi_{\xi}) - \underline{(\nabla\Phi_\xi\cdot\nabla)(\xi\psi_{(\xi)})}$$

notice
- $-\frac{1}{n_*^2\lambda^2}\psi_{(\xi)}\xi(\nabla\cdot\nabla\Phi_{(\xi)}) = -\psi_{(\xi)}\xi\,\Delta\Phi_{(\xi)} = \psi_{(\xi)}\xi\,\phi_{(\xi)} = W_{(\xi)}$.
- $(\psi_{(\xi)}\xi\cdot\nabla)(\nabla\Phi_{(\xi)}) = -\psi_{(\xi)}\xi\cdot\nabla(\nabla\Phi_{(\xi)}) = 0$.

b/c A_ξ, $\xi\times A_\xi$ comes out.

$$\Rightarrow w^{(c)}_{(\xi)} = \frac{1}{n_*^2\lambda^2}\operatorname{curl}(\Phi_{(\xi)}\operatorname{curl}(\psi_{(\xi)}\xi)) + \frac{1}{n_*^2\lambda^2}\nabla\psi_{(\xi)}\times\operatorname{curl}(\Phi_{(\xi)}\xi)$$

b/c. $\nabla\psi_{(\xi)}\times(\nabla\Phi_{(\xi)}\times\xi) = \underline{(\nabla\psi_{(\xi)}\cdot\xi)\nabla\Phi_{(\xi)}} - \underline{(\nabla\psi_{(\xi)}\cdot\nabla\Phi_{(\xi)})\xi}$.

Now notice for $\operatorname{curl}(\Phi_{(\xi)} \operatorname{curl}(\psi_{(\xi)} \xi)) = \operatorname{curl}(\Phi_{(\xi)} (\nabla\psi_{(\xi)} \times \xi))$

$= \Phi_{(\xi)} \nabla \times (\nabla\psi_{(\xi)} \times \xi) + \nabla\Phi_{(\xi)} \times (\nabla\psi_{(\xi)} \times \xi)$

But $\nabla\psi_{(\xi)}$ has ξ coming out.
and $\xi \times \xi = 0$. $\Rightarrow \operatorname{curl}(\psi_{(\xi)}\xi) = 0$.

Hence.

$$\operatorname{curl}\operatorname{curl} V_{(\xi)} - W_{(\xi)} = \boxed{W^{(c)}_{(\xi)}} \equiv \frac{1}{n_*^2\lambda^2} \nabla\psi_{(\xi)} \times \operatorname{curl}(\Phi_{(\xi)}\xi)$$

→ notice $\nabla\psi_{(\xi)}$ costs $\frac{r_\perp}{r_\parallel}\cdot\lambda$.
$\operatorname{curl}(\Phi_{(\xi)})$ costs λ

So $\|W^{(c)}_{(\xi)}\|_{L^p} \lesssim \frac{1}{\lambda^2} \cdot r_\parallel^{1/p-1/2} \frac{r_\perp \lambda}{r_\parallel} \cdot r_\perp^{2/p-1} \lambda$.

$\sim \frac{r_\perp}{r_\parallel} r_\parallel^{1/p-1/2} r_\perp^{2/p-1}$

in general. $\frac{r_\parallel}{r_\perp} \|\nabla^N \partial_t^M W^{(c)}_{(\xi)}\|_{L^p} \lesssim r_\perp^{2/p-1} r_\parallel^{1/p-1/2} \lambda^N \left(\frac{r_\perp \lambda}{r_\parallel}\mu\right)^M$

→ for $V_{(\xi)}$. we recover λ^2, then estimate.

$$\lambda^2 \|\nabla^N \partial_t^M V_{(\xi)}\|_{L^p} \lesssim r_\perp^{2/p-1} r_\parallel^{1/p-1/2} \lambda^N \left(\frac{r_\perp}{r_\parallel}\lambda\mu\right)^M$$

$\Rightarrow$ With these $\begin{cases} W_{(\xi)},\ W^{(c)}_{\xi},\ V_{(\xi)} \text{ in } L^p,\ W^{N,p} \ldots \left(\begin{matrix}p=\infty \text{ for}\\ \text{Hölder}\end{matrix}\right) \\ a_{(\xi)} \text{ in } C^N_{t,x}. \end{cases}$

6)

velocity increment & parameter choice

we obtain estimates for $w^{(p)}_{q+1}, w^{(c)}_{q+1}, w^{(t)}_{q+1}$

in both $\begin{cases} L^p \\ C^1_{t,x} \end{cases}$

$\Rightarrow$ we've done with estimating
$\|v_{q+1} - v_q\| \leq \delta_{q+1}^{1/2}$. , $\|v_{q+1}\|_{C^1_{t,x}} \leq \lambda_{q+1}^4$ $\|v_{q+1}\|_{L^2} \leq 1 - \delta_{q+1}$

provided we have good choice of parameters.

- Require i) $\ell^{-1} \boxed{\leq} \lambda_{q+1}^{2\alpha}$ for $0 < \alpha << 1$. auxiliary.

  ii) $\ell \lambda_q^4 \boxed{\lesssim} \lambda_{q+1}^{-\alpha} << \delta_{q+1}^{1/2}$.

  $\downarrow$ for $\beta < \alpha / b$.

  $\rightarrow$ choose $\ell \sim \lambda_{q+1}^{-3\alpha/2} \lambda_q^{-2}$. $\rightarrow$ this step needs $b > 4/\alpha$

i) is to deal with ℓ^{-1} brought by $\|a_{(\xi)}\|_{C^N}$.

ii) is to deal with

$$\begin{cases} \|\mathring{R}_{comm}\|_{L^1} \lesssim \ell \|v_q\|_{C^1} \|v_q\|_{C^0} \lesssim \ell \lambda_q^4 \lesssim \delta_{q+2} \\ \|v_q - v_\ell\|_{L^2} \lesssim \ell \|v_q\|_{C^1} \lesssim \ell \lambda_q^4 << \delta_{q+1}^{1/2} \\ \|v_\ell\|_{C^N_{t,x}} \lesssim \ell^{-N+1} \|v_q\|_{C^1} \lesssim \ell^{-N+1} \lambda_q^4 \end{cases}$$

$\hookrightarrow \|v_\ell\|_{C^1_{t,x}} \lesssim \lambda_q^4 \lesssim \ell^{-1} \lambda_{q+1}^{-\alpha} \lesssim \lambda_{q+1}^{\alpha}$

we want positive power of α for λ_{q+1} as α can be chosen small as we wish.

- Finally we make sense of $\lambda_{q+1}^{-1} << r_\perp << r_\parallel << 1$. by choosing.

$$\begin{cases} r_\parallel = \lambda_{q+1}^{-4/7} \\ r_\perp = \lambda_{q+1}^{-6/7} (2\pi)^{-1/4} \end{cases} \Rightarrow \lambda_{q+1} r_\perp = \lambda_{q+1}^{1/7} (2\pi)^{-1/7}$$

$$= a^{(b^{q+1})/7} \in \mathbb{N}$$

if we chose $7 \mid b$.

Recall for $\|\mathcal{R} \Delta w_{q+1}\|_{L^p}$ $p \in (1, 2]$ error.

$\lesssim \|\nabla w_{q+1}\|_{L^p}$

$\lesssim \|a\|_{C^1} \|W_{(\xi)}\|_{W^{1,p}}$

$\lesssim \lambda r_\perp^{2/p - 1} r_\parallel^{1/p - 1/2}$

Now if we choose $0 < \alpha \ll 1$ s.t.

$$r_\perp^{2\frac{1-p}{p}} r_{\|}^{\frac{1-p}{p}} \le (\lambda_{q+1}^{-6/7})^{2\frac{1-p}{p}} 2\pi^{-\frac{2}{7}\frac{1-p}{p}} \cdot \lambda_{q+1}^{-\frac{4}{7}\frac{1-p}{p}}$$

$$\le (2\pi)^{1/7} \lambda_{q+1}^{16\frac{p-1}{7p}} \le \lambda_{q+1}^{\alpha}$$

So $\|R_\Delta w_{q+1}\|_{L^p} \lesssim \|a\|_{C^1} \underbrace{r_\perp^{\frac{2}{p}-2} r_{\|}^{\frac{1}{p}-1}}_{\substack{\lesssim \lambda_{q+1}^{\alpha} \\ \text{small.}}} \cdot \underline{r_\perp r_{\|}^{1/2} \lambda}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \lambda_{q+1}^{-1/7}$

all errors should roughly remain at the level of $\lambda_{q+1}^{-1/7}$ after absorbing $r_\perp^{\frac{2}{p}-2} r_{\|}^{1/p-1}$.

Hence for $\|R\partial_t(w_{q+1}^{(p)} + w_{q+1}^{(c)})\|_{L^p}$

$$\lesssim \sum_{\xi\in\Lambda} \|\partial_t \operatorname{curl}(a_{(\xi)} V_{(\xi)})\|_{L^p}$$

$$\lesssim \sum_{\xi\in\Lambda} \left(\|a\|_{C^1} \boxed{\|\partial_t V_{(\xi)}\|_{W^{1,p}}} + \|\partial_t a\|_{C^1} \|V_{(\xi)}\|_{W^{1,p}} \right)$$

we need to control at level $\sim \lambda_{q+1}^{-1/7}$

i.e, $\|\nabla \partial_t V_{(\xi)}\|_{L^p} \lesssim \frac{1}{\lambda^2} r_\perp^{2/p-1} r_{\|}^{1/p-1/2} \lambda \cdot \frac{r_\perp}{r_{\|}} \lambda \mu$

$$= \underbrace{r_\perp^{2/p-2} r_{\|}^{1/p-1}}_{\lambda_{q+1}^{\alpha}} \underbrace{r_\perp^{2} r_{\|}^{-1/2} \mu}_{\text{should be of size } \lambda_{q+1}^{-1/7} \text{ or smaller.}}$$

Hence $\lambda_{q+1}^{-12/7} \lambda_{q+1}^{2/7} \mu = \lambda_{q+1}^{-10/7} \mu \triangleq \lambda_{q+1}^{-1/7}$.

$$\mu \sim \lambda_{q+1}^{9/7}$$

$$\mu := \lambda_{q+1} r_{\|} r_\perp^{-1} = \lambda_{q+1}^{9/7} (2\pi)^{1/7}$$

- How to choose a?

$$\|v_{q+1}\|_{L^2} \le \|v_\ell\|_{L^2} + \|w_{q+1}\|_{L^2}.$$

$$\lesssim 1 - \delta_q^{1/2} + \delta_{q+1}^{1/2}$$

take a so large s.t.

$$-\delta_q^{1/2} + \delta_{q+1}^{1/2} \lesssim -\delta_{q+1}^{1/2}.$$

$$\Rightarrow \|v_{q+1}\|_{L^2} \le 1 - \delta_{q+1}^{1/2} \text{ at } q+1 \text{ level.}$$

7) Finally we collect the Reynolds Stress Errors.

$$\operatorname{div} \mathring{R}_{q+1} - \nabla p_{q+1} = -\nu \Delta w_{q+1} + \partial_t (w_{q+1}^{(p)} + w_{q+1}^{(c)}) + \operatorname{div}(v_\ell \otimes w_{q+1} + w_{q+1} \otimes v_\ell)$$

$$\underset{\text{8 terms}}{+} \operatorname{div}\left((w_{q+1}^{(c)} + w_{q+1}^{(t)}) \otimes w_{q+1} + w_{q+1}^{(p)} \otimes (w_{q+1}^{(c)} + w_{q+1}^{(t)})\right)$$

$$\underset{\text{principal term}}{+} \operatorname{div}\left(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{R}_\ell\right) + \partial_t w_{q+1}^{(t)} + \operatorname{div}(R_{com}) - \nabla p_\ell$$

- $\underline{R_{linear}} = -\nu \mathcal{R} \Delta w_{q+1} + \mathcal{R} \partial_t (w_{q+1}^{(p)} + w_{q+1}^{(c)}) + v_\ell \mathring{\otimes} w_{q+1} + w_{q+1} \mathring{\otimes} v_\ell$

$$v_\ell \mathring{\otimes} w_{q+1} = v_\ell \otimes w_{q+1} - tr(v_\ell \otimes w_{q+1}) Id$$

$$\Updownarrow$$

$$v_\ell \cdot w_{q+1}.$$

pick $P_{linear} = 2 v_\ell \cdot w_{q+1}$

- $\underline{R_{corrector}} = (w_{q+1}^{(c)} + w_{q+1}^{(t)}) \mathring{\otimes} w_{q+1} + w_{q+1}^{(p)} \mathring{\otimes} (w_{q+1}^{(c)} + w_{q+1}^{(t)})$

pick. $P_{corrector} = |w_{q+1}|^2 - |w_{q+1}^{(p)}|^2.$

- $\underline{R_{oscillation}} = \underbrace{\sum_{\xi \in \Lambda} \mathcal{R}\left(\nabla a_{(\xi)}^2 \mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)})\right)}_{R^{(x)}_{oscillation}} - \underbrace{\frac{1}{\mu} \sum_{\xi \in \Lambda} \mathcal{R}\left(\partial_t a_{(\xi)}^2 (\phi_{(\xi)}^2 \psi_{(\xi)}^2 \xi)\right)}_{R^{(t)}_{oscillation}}.$

pick $P_{oscillation} = \rho + P.$

$$\Rightarrow \begin{cases} \mathring{R}_{q+1} = R_{linear} + R_{cor.} + R_{osc} + R_{com}. \\ p_{q+1} = p_\ell - p_{osc} - p_{cor} - p_{linear}. \end{cases}$$

A trick to compute for $R^{(x)}_{\text{oscillation}} = \sum_{\xi \in \Lambda} R(\nabla a^2_{(\xi)} \mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)}))$

since $W_{(\xi)} \otimes W_{(\xi)}$ $(\pi/\lambda_{q+1} r_\perp)^3$ periodic.

$\mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)}) = \mathbb{P}_{\geq r_\perp \lambda_{q+1}}(W_{(\xi)} \otimes W_{(\xi)})$. high frequency

yet $\nabla a^2_{(\xi)}$ oscillates at much lower freq.

we have lemma $\| |\nabla|^{-1} (a \mathbb{P}_{\geq \kappa} f) \|_{L^p} \lesssim C_a \frac{\|f\|_{L^p}}{\kappa}$. $p \in (1, 2]$

for $\|D^j a\|_{C^0} \leq C_a \zeta^j$

here $C_a = \ell^{-9}$. $\kappa = \lambda_{q+1} r_\perp$. $f = W_{(\xi)} \otimes W_{(\xi)}$.

$\Rightarrow \|R^{(x)}_{\text{oscillation}}\|_{L^p} \leq \ell^{-9} \frac{\|W_{(\xi)} \otimes W_{(\xi)}\|_{L^p}}{\lambda_{q+1} r_\perp}$

$\leq \ell^{-9} \frac{1}{\lambda_{q+1} r_\perp} \|W_{(\xi)}\|^2_{L^{2p}}$

$\lesssim \ell^{-9} \lambda^{\alpha}_{q+1} (r_\perp^{-1} \lambda^{-1}_{q+1})$

$\lambda_{q+1}^{4/7 - 1} = \lambda_{q+1}^{-3/7}$

we're happy.

- Why $p \in (1, 2]$?

∇R fail to be bounded on L^1.

$\|\mathring{R}_{q+1}\|_{L^1} \lesssim \|R_{\text{linear}}\|_{L^p} + \|R_{\text{corrector}}\|_{L^p} + \|R_{\text{oscillation}}\|_{L^p} + \|\mathring{R}_{\text{ℓ}}\|_{L^1}$.

$\lesssim \lambda_{q+1}^{29\alpha - 1/7}$ our choice of error level $+ \lambda_{q+1}^{-\alpha}$

$\leq \lambda_{q+1}^{30\alpha - 1/7} + \lambda_{q+1}^{-\alpha/2}$

use $\beta b < \alpha$. so $\delta_{q+2} = \lambda_{q+2}^{-2\beta} \sim a^{(b^{q+2}) \cdot (-2\beta)}$

$= a^{(b^{q+1})(-2\beta b)}$

$\Rightarrow \lesssim \delta_{q+2}$. $> a^{(b^{q+1}) \alpha / 2}$

$\sim \lambda_{q+1}^{-\alpha/2}$

8). We've proved the Proposition.

now we prove our theorem by the iteration

Define $V_0(x,t) = \frac{t}{(2\pi)^{3/2}} \begin{pmatrix} \sin(\lambda_0 x_3) \\ 0 \\ 0 \end{pmatrix}$.

observe V_0 is shear flow.

$$\begin{cases} (V_0 \cdot \nabla) V_0 = \frac{t}{(2\pi)^{3/2}} \begin{pmatrix} \sin(\lambda_0 x_3) \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \frac{t\lambda_0}{(2\pi)^{3/2}} \cos(\lambda_0 x_3) \end{pmatrix} = 0 \\ \nabla \cdot V_0 = 0 \quad \text{incompressible.} \end{cases}$$

compute. $\partial_t V_0 - \nu V_0 = \frac{1}{(2\pi)^{3/2}} (1 + \nu \lambda_0^2 t) \begin{pmatrix} \sin(\lambda_0 x_3) \\ 0 \\ 0 \end{pmatrix} = (*)$

estimate.

- $$\sup_{t\in[0,1]} \|V_0(\cdot,t)\|_{L^2} \le \|V_0(\cdot,1)\|_{L^2}$$
$$= \left(\frac{1}{(2\pi)^3} \int_{x_3} \sin^2(\lambda_0 x_3)\, dx_3 \, (2\pi)^2 \right)^{1/2}$$
$$= \left(\frac{1}{2\pi} \int_{x_3} \frac{1}{2} (1 + \cos(2\lambda_0 x_3))\, dx_3 \right)^{1/2}$$
$$\le \frac{1}{\sqrt{2}} \le 1 - \delta_0^{1/2}$$

at this stage take $a \nearrow$ so that $\delta_0 \le 1/4$

- $\|V_0\|_{C^1_{x,t}} \le \lambda_0 \le \lambda_0^4$ b/c $\lambda_0 = 2\pi a > 1$.

Also define $\mathring{R}_0 = -\frac{1+\nu t\lambda_0^2}{\lambda_0 (2\pi)^{3/2}} \begin{pmatrix} 0 & 0 & \cos(\lambda_0 x_3) \\ 0 & 0 & 0 \\ \cos(\lambda_0 x_3) & 0 & 0 \end{pmatrix}$

so $\operatorname{div} \mathring{R}_0 = -\frac{1+\nu t\lambda_0^2}{\lambda_0 (2\pi)^{3/2}} \begin{pmatrix} -\lambda_0 \sin(\lambda_0 x_3) \\ 0 \\ 0 \end{pmatrix} = (*)$

So $\partial_t V_0 + \underbrace{\operatorname{div}(V_0 \otimes V_0)}_{=0} - \nu V_0 = \operatorname{div} \mathring{R}_0$

- Now we take $\nu = \lambda_0^{-2} \in (0, 1]$.

$$\|\mathring{R}_0\|_{L^1} \le \frac{C}{\lambda_0} \le \delta_1$$

b/c $\lambda_0 \delta_1 = 2\pi a \cdot (2\pi a^b)^{-2\beta} = 2\pi^{1-2\beta} a^{1-2b\beta}$.

since $\beta b \le 1/4$ $\Rightarrow$ $\lambda_0 \delta_1 \ge a^{1/2} \ge C$

$\Rightarrow$ we start iteration.

Define $v := \sum_{q \ge 0} (v_{q+1} - v_q) + v_0$, $\forall \beta' \in (0, \frac{\beta}{4+\beta})$

$$\|v\|_{C_t^0 H_x^{\beta'}} \le \sum_{q \ge 0} \|v_{q+1} - v_q\|_{H^{\beta'}}$$

$$\overset{\text{Hölder}}{\le} \sum_{q \ge 0} \underbrace{\|v_{q+1} - v_q\|_{L^2}^{1-\beta'}}_{\text{given by iteration}} \underbrace{\|v_{q+1} - v_q\|^{\beta'}_{H^1}}_{\text{given by } C^1_{t,x}}$$

$$\le \sum_{q \ge 0} \delta_{q+1}^{\frac{1-\beta'}{2}} \lambda_{q+1}^{4\beta'} \le \sum_{q \ge 0} \lambda_{q+1}^{-\beta + \beta\beta' + 4\beta'}$$

since $\beta'(4+\beta) = 4\beta' + \beta\beta' < \beta$. series converges.

$\Rightarrow$ $v \in C_t^0 H_x^{\beta'} \subset C_t^0 L_x^2$.

Why v solves as mild weak sol? $\|\mathring{R}_q\|_{L^1} \to 0$ as $q \to \infty$.

By semigroup maximal regularity of heat eq. $v \in C_t^0 W^{1, 1+\beta''}(\mathbb{T}^3)$
$\beta'' \in (0, \beta')$.

Remains to show $\|v(\cdot,1)\|_{L^2} > 2\|v(\cdot,0)\|_{L^2}$.

- note $\sup_{t\in[0,1]} \|v - v_0\|_{L^2} \le \sum_{q\ge 0} \|v_{q+1} - v_q\|_{L^2} \le \sum_{q\ge 0} \delta_{q+1}^{1/2}$

$$\le \sum_q \lambda_{q+1}^{-\beta} = \sum_q (2\pi)^{-\beta} a^{-\beta b^{q+1}}.$$

use $b^{q+1} \ge b(q+1)$ for b, q large.

$$\lesssim \sum_{q\ge 0} a^{-\beta b(q+1)}$$

$$= \frac{a^{-\beta b}}{1 - a^{-\beta b}} \le \frac{1}{6}$$

choice of a large w.r.t. β, b

- also Recall. $\|v_0(\cdot,1)\|_{L^2} = \frac{1}{\sqrt{2}}$ $\|v_0(\cdot,0)\|_{L^2} = 0$.

we have $2\|v(\cdot,0)\|_{L^2} \le 2\left(\|v_0(\cdot,0)\|_{L^2} + \|v(\cdot,0) - v_0(\cdot,0)\|_{L^2}\right)$

$$\le 2\cdot\frac{1}{6} = \frac{1}{3} < \frac{1}{\sqrt{2}} - \frac{1}{6}$$

$$\le \|v_0(\cdot,1)\|_{L^2} - \|v(\cdot,1) - v_0(\cdot,1)\|_{L^2}$$

$$\le \|v(\cdot,1)\|_{L^2}.$$

$\square$.